Then, rad $C = C \cap$ rad R. Therefore, we have a k - embedding $\varphi : C/\text{rad } C \to R/\text{rad } R$. Since R splits over k, $R/\text{rad } R \sim \prod_{i=1}^{r} M_{n_i}(k)$ for suitable $n_1, \ldots, n_r$. We have then $\varphi(C/\text{rad } C) \subseteq Z(R/\text{rad } R) \sim \prod_{i=1}^{r} k$. We claim that every k-subalgebra of $B := \prod_{i=1}^{r} k$ *is k-isomorphic to* $\prod_{i=1}^{s} k$ for some $s \leq r$. Assuming this claim, we will have $C/\text{rad } C \sim \varphi(C/\text{rad } C) \sim \prod_{i=1}^{s} k$ so, C splits over k. To prove our claim, consider a k-subalgebra $A \subseteq B$. Since A is commutative, reduced, and artinian, $A = K1 \times \cdots \times Ks$ for suitable finite field extensions Ki/k. We finish by showing that $Ki = k$ for all i. Let ei be the identity of Ki. For a suitable coordinate projection π of $B = \prod_{i=1}^{r} k$ onto k, we have $\pi(ei) \neq 0$. Since $\pi(ei)$ is an idempotent, we must have $\pi(ei) = 1$. Thus, π defines a k-algebra homomorphism from Ki to k, and it follows that $\pi : Ki \sim k$, as desired.
