First suppose $R = kG$ is von Neumann regular. As a quotient ring of R, k is certainly von Neumann regular. Consider any finite subgroup $E \subseteq G$, say of order m. It suffices to show that any prime $p \mid m$ is a unit in k. To see this, we fix an element $\sigma \in E$ of order p (which exists by Cauchy's Theorem). Taking an element $\alpha \in R$ such that $1 - \sigma = (1 - \sigma)\alpha(1 - \sigma)$, we can argue that $p \in \mathrm{U}(k)$. Finally, let H be a subgroup of G generated by a finite set of elements, say $h_1, \ldots, h_n$. It is easy to see that

$$I := \sum_{h \exists \in H} R \cdot (h-1) = \sum_{i=1}^{n} R \cdot (h_i - 1)$$

(using facts such as $h_i^{-1} - 1 = h_i^{-1}(1 - h_i)$, and $h_i h_j - 1 = h_i(h_j - 1) + h_i - 1$). Since R is von Neumann regular, $I = Re$ for some idempotent e. Now I is contained in the augmentation ideal of R, so $e \neq 1$ (since $k \neq 0$).
Thus, $f := 1 - e \neq 0$. But for any $h \in H$, $(h - 1)f \in I \cdot f = R \cdot ef = 0 \Rightarrow f = hf$. Since f involves only finitely many elements of G, this implies that $|H| < \infty$.
For the converse, let us assume the given conditions on k and G. Consider any element $\alpha = a_1 h_1 + \cdots + a_n h_n \in R$ ($a_i \in k$, $h_i \in G$), for which we want to show $\alpha \in \alpha R \alpha$. Let H be the (finite) subgroup of G generated by $h_1, \ldots, h_n$. Since $\alpha \in kH$, we are done if we can show that $S := kH$ is von Neumann regular. Consider any principal left ideal, say $S \cdot \beta$, where $\beta \in S$. Viewing $S \cdot \beta \subseteq S$ as k-modules, we have $S \sim k^{|H|}$ and $S \cdot \beta = \sum_{h \in H} k \cdot (h\beta)$. Since k is von Neumann regular, $S \cdot \beta$ is a direct summand of S *as* k – modules. Then $S \cdot \beta$ is a direct summand of S as S-modules. This checks that S is von Neumann regular, as desired.