Let $H_r = \left\langle y^{3^r} \right\rangle \lhd G$, $\left(1 \le r < \infty\right)$ and let $G_r = G/H_r$, which is a dihedral group of order $2 \cdot 3^r$. We first show that the natural ring homomorphism $\varphi = (\varphi_r) : kG \to \prod_r kG_r$ is *injective*. Indeed, consider a nonzero element

$\alpha = a_1g_1 + \cdots + a_ng_n \in R$,

where a_i are nonzero for all i, and $g_1, \ldots, g_n$ are distinct elements of G. Since $H_1 \supseteq H_2 \supseteq \cdots$ and $\bigcap H_r = \{1\}$, there exists an s such that $g_i^{-1}g_j \notin H_s$ for all $i \neq j$. But then $g_iH_s \neq g_jH_s$, and we have

$\varphi_s(\alpha) = a_1g_1H_s + \cdots + a_ng_nH_s \neq 0 \in kG_s$.

Now, $(\text{rad } kG_r)^2 = 0$ for all r. From $\varphi_r(\text{rad } kG) \subseteq \text{rad } kG_r$, we have $\varphi_r((\text{rad } kG)^2) = 0$. Since this holds for all r, the injectivity of φ yields $(\text{rad } kG)^2 = 0$. But then $\text{rad } kG = 0$, as desired.