We khow that $J = \ker(\varphi) = \{x + y(123) + z(132) + u(12) + v(13) + w(23) : x + y + z = u + v + w = 0\}$.
A k-basis for J is therefore given by
$\alpha_1 = (123) - 1$, $\alpha_2 = (132) - 1$, $\alpha_3 = (12) - (13)$, $\alpha_4 = (12) - (23)$.
A simple computation shows that each product $\alpha_i\alpha_j$ is of the form $\pm\beta_1$ or $\pm\beta_2$, where $\beta_1 = 1 + (123) + (132)$, and $\beta_2 = (12) + (13) + (23)$. Thus, J^2 has k-basis $\{\beta_1, \beta_2\}$. By inspection, we see that $\alpha_i\beta_j = 0$ for all i, j. Therefore, $J^3 = 0$, so the index of nilpotency for J is 3.