We may assume that char $k = p > 0$, for otherwise both sides of the equation are zero. Let F_p denote the prime field of k. We shall first prove the following special case: $\text{rad}(kG) = (\text{rad } F_pG) \otimes_{F_p} k$.
$(F_pG/\text{rad } F_pG) \otimes_{F_p} k \sim kG/ ((\text{rad } F_pG) \otimes_{F_p} k)$ is semisimple. Therefore, we must have $\text{rad } kG \subseteq (\text{rad } F_pG) \otimes_{F_p} k$. Since the latter is a nilpotent ideal in kG, equality holds. This proves special case, and we deduce immediately that $\text{rad}(KG) = (\text{rad } F_pG) \otimes_{F_p} K = (\text{rad } F_pG) \otimes_{F_p} k \otimes_k K = (\text{rad } kG) \otimes_k K$.