Give an example to show, that the sum of two semiprime ideals need not be semiprime.

For $A, B \in S$, we have $A \cap B \in S$. Thus, $inf\{A, B\}$ is given simply by $A \cap B$. For $sup\{A, B\}$, we take $\sqrt{A+B} \in S$. A semiprime ideal $C$ contains both $A$ and $B$ iff $C \supseteq \sqrt{A+B}$. Thus, $\sqrt{A+B}$ is indeed the supremum of $A$ and $B$ in $S$. This shows that $S$ is a lattice. Clearly, $S$ has a largest element, $R$, and a smallest element, $Nil_* R$.

In the above construction, we cannot replace $\sqrt{A+B}$ by $A+B$, since $A+B$ may not be semiprime. For an explicit example of this, consider $R = \mathbb{Z}[x]$, in which $A = (x)$ and $B = (x-4)$ are (semi)prime ideals (since $R/A \cong \mathbb{Z} \cong R/B$). Here,

$$A + B = (x, x-4) = (x, 4)$$

Is not semiprime (since $R/(A+B) \cong \mathbb{Z}_4$), and we have

$$sup\{A, B\} = \sqrt{A+B} = (2, x).$$

Alternatively, we could have also taken $A = (2)$ and $B = (x^2 - 2)$, for which $A + B = (2, x^2)$, is not semiprime. Here $sup\{A, B\}$ is again $(2, x)$.

In spite of these examples, there are many rings in which we do have $sup\{A, B\} = A + B$ for semiprime ideals $A$ and $B$. These include, for instance, von Neumann regular rings, and left (right) artinian rings, as you can easily verify. The ring $\mathbb{Z}$ is another example: here, $A + B$ is semiprime as long as $one\ of$ $A, B$ is semiprime!

*Comment.* The $S$ in this exercise is actually a complete lattice, in the sense that "sup" and "inf" exist for arbitrary subsets in $S$. If $\{A_i : i \in I\} \subseteq S$, the infimum is given as before by the semiprime ideal $\cap_i A_i \in S$, and the supremum is given by the semiprime ideal $\sqrt{\sum_i A_i}$.