Question 1. *Prove that if $AB = BA$ for every N by N matrix A, then $B = cI$ for some constant c.*

Solution. It follows from $AB = BA$ that A should also be square of size N. For each $i = 1, \ldots, N$ consider the diagonal matrix C_i, whose (i,i)-th entry is 1 and all the other entries are zero. Since $AC_i = C_iA$, then for all $j \neq i$

$$(AC_i)(i,j) = (C_iA)(i,j) \Leftrightarrow 0 = 1 \cdot A(i,j) \Leftrightarrow A(i,j) = 0.$$

Thus, we proved that A is diagonal. Now show that $A(i,i) = A(j,j)$ for all $1 \leq i,j \leq N$. Consider the matrix D_{ij}, whose unique nonzero entry is $D_{ij}(i,j) = 1$. Then $AD_{ij} = D_{ij}A$ implies

$$(AD_{ij})(i,j) = (D_{ij}A)(i,j) \Leftrightarrow A(i,i) \cdot D_{ij}(i,j) = D_{ij}(i,j) \cdot A(j,j) \Leftrightarrow A(i,i) = A(j,j).$$

So, A is diagonal and all the diagonal entries of A are equal, i. e. $A = cI$ for some constant c. □