**Question 1.** *Show that, in the vector space $V = \mathbb{R}^2$, the subset of all vectors whose entries sum to zero is a subspace, but the subset of all vectors whose entries sum to one is not a subspace.*

*Solution.* Denote

$$A = \{v = (x, y) \in V \mid x + y = 0\}$$

and

$$B = \{v = (x, y) \in V \mid x + y = 1\}.$$

Prove that $A$ is a subspace of $V$, while $B$ is not a subspace.

Indeed, for any two vectors $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ from $A$ and for any scalars $\alpha_1, \alpha_2 \in \mathbb{R}$ consider the linear combination

$$\alpha_1 v_1 + \alpha_2 v_2 = (\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2).$$

Since

$$(\alpha_1 x_1 + \alpha_2 x_2) + (\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1(x_1 + y_1) + \alpha_2(x_2 + y_2) = \alpha_1 \cdot 0 + \alpha_2 \cdot 0 = 0,$$

we conclude that $\alpha_1 v_1 + \alpha_2 v_2 \in A$. This shows that $A$ is a subspace of $V$.

The fact that $B$ is not a subspace is obvious, because the zero vector $(0, 0)$ does not belong to $B$: the sum of its coordinates is $0 \neq 1$. $\square$