Given that cosec x + cot x = 3, evaluate the following:
(i) cosec x - cot x
(ii) cos x

Using the definitions of the trigonometric functions

$cosecx = \frac{1}{sinx}$

$cotx = \frac{cosx}{sinx}$

$$\frac{1}{sinx} + \frac{cosx}{sinx} = 3$$

Multiplying both sides by $sinx$ $(sinx \neq 0)$

$1 + cosx = 3sinx$

$1 + cosx - 3sinx = 0$

Using the double-angle formulae

$$sinx = 2sin\frac{x}{2}cos\frac{x}{2}$$

$cosx = 2cos^2\frac{x}{2} - 1$ ($cosx + 1 = 2cos^2\frac{x}{2}$)

$$2cos^2\frac{x}{2} - 6sin\frac{x}{2}cos\frac{x}{2} = 0$$

Solving the equation $cos\frac{x}{2}\left(cos\frac{x}{2} - 3sin\frac{x}{2}\right) = 0$ we get

1) $cos\frac{x}{2} = 0$ (is not a solution because $sinx = 2sin\frac{x}{2}cos\frac{x}{2} \neq 0$)

2) $cos\frac{x}{2} - 3sin\frac{x}{2} = 0$

Dividing by $cos\frac{x}{2}$ we get

$1 - 3tg\frac{x}{2} = 0$, so $tg\frac{x}{2} = \frac{1}{3}$

Using the double-angle formulae

$$sinx = 2sin\frac{x}{2}cos\frac{x}{2}$$

$$cosx = 2cos^2\frac{x}{2} - 1 = 1 - 2sin^2\frac{x}{2}$$

$$cos^2\frac{x}{2} = \frac{1 + cosx}{2}$$

$$sin^2\frac{x}{2} = \frac{1 - cosx}{2}$$

i.

$$cosecx - cotx = \frac{1}{2sin\frac{x}{2}cos\frac{x}{2}} - \frac{1-2sin^2\frac{x}{2}}{2sin\frac{x}{2}cos\frac{x}{2}} = \frac{2sin^2\frac{x}{2}}{2sin\frac{x}{2}cos\frac{x}{2}} = \frac{sin\frac{x}{2}}{cos\frac{x}{2}} = tg\frac{x}{2} = \frac{1}{3}$$

ii.

$$tg^2x = \frac{1-cosx}{1+cosx}$$

Hence

$$cosx = \frac{1-tg^2x}{1+tg^2x} = \frac{1-\frac{1}{9}}{1+\frac{1}{9}} = \frac{4}{5}$$

Answer: i. $cosecx - cotx = \frac{1}{3}$ ii. $cosx = \frac{4}{5}$