Use Lagrange multipliers to solve the given optimization problem. Find the minimum value of $f(x, y) = 4xy$ subject to $x^2 + y^2 = 8$. Also find the corresponding point(s)

Solution:

The method of Lagrange multipliers allows us to maximize or minimize functions with the constraint that we only consider points on a certain surface. To find critical points of a function $f(x, y)$ on a level surface $g(x, y) = C$, we must solve the following system of simultaneous equations:

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

$$g(x, y) = C$$

The variable λ is a dummy variable called a Lagrange multiplier; we only really care about the values of x, y.

We have $f(x, y) = 4xy$ and $g(x, y) = x^2 + y^2$, $C = 8$

$$\frac{\partial f}{\partial x} = 4y \qquad \frac{\partial f}{\partial y} = 4x \qquad \frac{\partial g}{\partial x} = 2x \qquad \frac{\partial g}{\partial y} = 2y$$

The equations to be solved are thus

$$4y = \lambda * 2x$$

$$4x = \lambda * 2y$$

$$x^2 + y^2 = 8$$

From the first equation

$$\begin{cases} \lambda = \frac{2y}{x} \\ 4x = \frac{4y^2}{x} \\ x^2 + y^2 = 8 \end{cases}$$

$$\begin{cases} \lambda = \frac{2y}{x} \\ x^2 = y^2 \\ y^2 + y^2 = 8 \end{cases}$$

$$\begin{cases} \lambda = \frac{2y}{x} \\ x^2 = y^2 \\ y^2 = 4 \end{cases}$$

$$\begin{cases} \lambda = \dfrac{2y}{x} \\ x = \pm 2 \\ y = \pm 2 \end{cases}$$

There are thus four critical points: $(-2,-2)$, $(-2,2)$, $(2,-2)$, $(2,2)$;

Since $f(x,y) = 4xy$ we have

$$f(-2,-2) = f(2,2) = 16$$

$$f(-2,2) = f(2,-2) = -16$$

Thus $(-2,-2)$ and $(2,2)$ are maxima and $(-2,2)$ and $(2,-2)$ are minima

Answer: minimum value $f(x,y) = -16$ at the points $(-2,2)$ and $(2,-2)$