Question 1. *Let $\phi : \mathbb{R} \to \mathbb{R}^*$, where $\mathbb{R}$ is additive and $\mathbb{R}^*$ is multiplicative, be defined by $\phi(x) = 2^x$. Prove that ϕ is an isomorphism or not.*

Solution. Recall that $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is a multiplicative subgroup of the field $\mathbb{R}$. First of all note that ϕ is defined correctly, i. e. $\phi(x) \in \mathbb{R}^*$ for all $x \in \mathbb{R}$. Indeed, $\phi(x) = 2^x \neq 0$ for all $x \in \mathbb{R}$. Furthermore, note that ϕ is a homomorphism, because

$$\phi(x + y) = 2^{x+y} = 2^x \cdot 2^y = \phi(x)\phi(y).$$

Moreover, ϕ is a monomorphism, as

$$\phi(x) = \phi(y) \Leftrightarrow 2^x = 2^y \Leftrightarrow \log_2(2^x) = \log_2(2^y) \Leftrightarrow x = y.$$

But ϕ is not surjective, because $\phi(x) = 2^x > 0$, so ϕ maps $\mathbb{R}$ onto the group $\mathbb{R}^+$ of positive real numbers, but not onto the group $\mathbb{R}^*$ of all nonzero real numbers. Thus, ϕ is not isomorphism, but just a monomorphism.
Answer: it is not an isomorphism, but a monomorphism. □