Question 1. *Let F be an additive group of all continuous functions mapping $\mathbb{R}$ into $\mathbb{R}$. Let $\mathbb{R}$ be the additive group of real numbers, and let $\phi : F \to \mathbb{R}$ be given by $\phi(f) = \int_0^4 f(x)dx$. Prove that f is a homomorphism.*

Solution. Recall that the addition of any two $f, g \in F$ is defined coordinate-wise:

$$(f + g)(x) = f(x) + g(x).$$

Now taking arbitrary $f, g \in F$ by the additivity of integration we note that

$$\begin{aligned} \phi(f + g) &= \int_0^4 (f + g)(x)dx \\ &= \int_0^4 (f(x) + g(x))dx \\ &= \int_0^4 f(x)dx + \int_0^4 g(x)dx \\ &= \phi(f) + \phi(g). \end{aligned}$$

Thus, ϕ is a homomorphism from F to $\mathbb{R}$. □