Question 1. *Let J be a nilpotent right ideal in a ring R. If I is a subgroup of J such that $I \cdot I \subseteq I$ and $J = I + J^2$, show that $I = J$.*

Solution. Prove by induction that $J = I + J^n$ for all $n \geq 2$. The base of induction is given: $J = I + J^2$. The inductive step: suppose $J = I + J^n$ and prove that $J = I + J^{n+1}$. We have

$$J^2 = J \cdot J = J(I + J^n) = JI + J^{n+1} = (I + J^n)I + J^{n+1} = I \cdot I + J^n I + J^{n+1}.$$

It is given that $I \cdot I \subseteq I$. Furthermore, $J^n I \subset J^n \cdot J = J^{n+1}$, because $I \subseteq J$. Finally, $J^{n+1} + J^{n+1} \subseteq J^{n+1}$, as any right ideal is closed under addition. Thus,

$$J^2 = I \cdot I + J^n I + J^{n+1} \subseteq I + J^{n+1} + J^{n+1} = I + J^{n+1}.$$

Therefore,

$$J = I + J^2 \subseteq I + (I + J^{n+1}) = (I + I) + J^{n+1} = I + J^{n+1},$$

because I is an additive subgroup of J. The converse inclusion $I + J^{n+1} \subseteq J$ is obvious, because J is closed under addition and multiplication and $I \subseteq J$. So, $J = I + J^{n+1}$.

Since J is nilpotent, there is $n \geq 2$ such that $J^n = 0$. Taking this n, we get

$$J = I + J^n = I + 0 = I.$$

□