Question 1. *If R is a ring with identity, then prove that*

(a) $\langle a\rangle_l = \{ta \mid t \in R\} = Ra$;

(b) $\langle a\rangle_r = \{at \mid t \in R\} = aR$;

(c) $\langle a\rangle = \{\sum_{i=1}^n s_i a t_i \mid n \in \mathbb{N}, s_i, t_i \in R\} = RaR$.

Solution. (a) Prove that Ra is a left ideal. Indeed, for any $ta \in Ra$ and for any $r \in R$ we have

$$r \cdot ta = rt \cdot a \in Ra.$$

Furthermore, for arbitrary $t_1 a, t_2 a \in Ra$:

$$t_1 a - t_2 a = (t_1 - t_2)a \in Ra.$$

So, Ra is a subgroup of additive group of R. Obviously, Ra contains a, because $a = 1 \cdot a$. So, $\langle a\rangle_l$, as a minimal left ideal containing a, is a subset of Ra. Conversely, since $a \in \langle a\rangle_l$ and $\langle a\rangle_l$ is a left ideal by definition, then for any $t \in R$ the product ta should belong to $\langle a\rangle_l$. Hence, $Ra \subset \langle a\rangle_l$. Both inclusions mean that $\langle a\rangle_l = \{ta \mid t \in R\} = Ra$.

(b) It is proved similarly to (a) by replacing "left" with "right".

(c) As above first of all show that RaR is an ideal of R. Taking any $\sum_{i=1}^n s_i a t_i \in RaR$ and multiplying it on the left or on the right by $r \in R$ we get

$$\sum_{i=1}^n r s_i a t_i \in RaR, \quad \sum_{i=1}^n s_i a t_i r \in RaR.$$

Now for arbitrary $\sum_{i=1}^n s_i a t_i \in RaR$ and $\sum_{j=1}^m u_j a v_j \in RaR$ their difference is

$$\sum_{k=1}^{m+n} x_k a y_k \in RaR,$$

where

$$x_k = \begin{cases} s_k, & 1 \le k \le n, \\ -u_{k-n}, & n+1 \le k \le m+n \end{cases}, \quad y_k = \begin{cases} t_k, & 1 \le k \le n, \\ -v_{k-n}, & n+1 \le k \le m+n \end{cases}.$$

Hence RaR is a subgroup of additive group of R. The converse inclusion: since $a \in \langle a\rangle$ and $\langle a\rangle$ is a (two-sided) ideal, then for all $s_i, t_i \in R$, $i = 1, \dots, n$, we have $s_i a t_i \in R$. Therefore, $\sum_{i=1}^n s_i a t_i \in \langle a\rangle$, because an ideal is closed under addition. Thus, $RaR \subset \langle a\rangle$. □