Conditions

prove Principle of Mathematical Induction using Peano axioms

Solution

The Peano axioms define the arithmetical properties of natural numbers, usually represented as a set N or The signature (a formal language's non-logical symbols) for the axioms includes a constant symbol 0 and a unary function symbol S.

The constant 0 is assumed to be a natural number:

0 is a natural number.

The next four axioms describe the equality relation.

For every natural number x, x = x. That is, equality is reflexive.

For all natural numbers x and y, if x = y, then y = x. That is, equality is symmetric.

For all natural numbers x, y and z, if x = y and y = z, then x = z. That is, equality is transitive.

For all a and b, if a is a natural number and a = b, then b is also a natural number. That is, the natural numbers are closed under equality.

The remaining axioms define the arithmetical properties of the natural numbers. The naturals are assumed to be closed under a single-valued "successor" function S.

For every natural number n, S(n) is a natural number.

Peano's original formulation of the axioms used 1 instead of 0 as the "first" natural number. This choice is arbitrary, as axiom 1 does not endow the constant 0 with any additional properties. However, because 0 is the additive identity in arithmetic, most modern formulations of the Peano axioms start from 0. Axioms 1 and 6 define a unary representation of the natural numbers: the number 1 can be defined as S(0), 2 as S(S(0)) (which is also S(1)), and, in general, any natural number n as Sn(0). The next two axioms define the properties of this representation.

For every natural number n, S(n) = 0 is false. That is, there is no natural number whose successor is 0.

For all natural numbers m and n, if S(m) = S(n), then m = n. That is, S is an injection.

Axioms 1, 6, 7 and 8 imply that the set of natural numbers contains the distinct elements 0, S(0), S(S(0)), and furthermore that {0, S(0), S(S(0)), …} ⊆ N. This shows that the set of natural numbers is infinite. However, to show that N = {0, S(0), S(S(0)), …}, it must be shown that N ⊆ {0, S(0), S(S(0)), …}; i.e., it must be shown that every natural number is included in {0, S(0), S(S(0)), …}. To do this however requires an additional axiom, which is sometimes called the axiom of induction. This axiom provides a method for reasoning about the set of all natural numbers.

If K is a set such that:

0 is in K, and

for every natural number n, if n is in K, then S(n) is in K,

then K contains every natural number.

The induction axiom is sometimes stated in the following form:

If ϕ is a unary predicate such that:

$\phi(0)$ is true, and

for every natural number n, if $\phi(n)$ is true, then $\phi(S(n))$ is true,

then $\phi(n)$ is true for every natural number n.

Now let's prove the principle of mathematical induction, using these axioms.

For instance, it can be proved if one assumes:

The set of natural numbers is well-ordered.

Every natural number is either zero, or n+1 for some natural number n.

For any natural number n, n+1 is greater than n.

To derive simple induction from these axioms, we must show that if P(n) is some proposition predicated of n, and if:

P(0) holds and

whenever P(k) is true then P(k+1) is also true

then P(n) holds for all n.

Proof. Let S be the set of all natural numbers for which P(n) is false. Let us see what happens if we assert that S is nonempty. Well-ordering tells us that S has a least element, say t. Moreover, since P(0) is true, t is not 0. Since every natural number is either zero or some n+1, there is some natural number n such that n+1=t. Now n is less than t, and t is the least element of S. It follows that n is not in S, and so P(n) is true. This means that P(n+1) is true, and so P(t) is true. This is a contradiction, since t was in S. Therefore, S is empty.

It can also be proved that induction, given the other axioms, implies well-ordering.