**Question 1.** *If $R$ is a ring with identity, then prove that* $\langle a\rangle = \{\sum_{i=1}^n s_i a t_i \mid n \in \mathbb{N}, s_i, t_i \in R\} = RaR$.

*Solution.* First of all show that $RaR$ is an ideal of $R$. Taking any $\sum_{i=1}^n s_i a t_i \in RaR$ and multiplying it on the left or on the right by $r \in R$ we get

$$\sum_{i=1}^n r s_i a t_i \in RaR, \quad \sum_{i=1}^n s_i a t_i r \in RaR.$$

Now for arbitrary $\sum_{i=1}^n s_i a t_i \in RaR$ and $\sum_{j=1}^m u_j a v_j \in RaR$ their difference is

$$\sum_{k=1}^{m+n} x_k a y_k \in RaR,$$

where

$$x_k = \begin{cases} s_k, & 1 \le k \le n, \\ -u_{k-n}, & n+1 \le k \le m+n \end{cases}, \quad y_k = \begin{cases} t_k, & 1 \le k \le n, \\ -v_{k-n}, & n+1 \le k \le m+n \end{cases}.$$

Hence $RaR$ is a subgroup of additive group of $R$. So, $\langle a\rangle \subset RaR$. The converse inclusion: since $a \in \langle a\rangle$ and $\langle a\rangle$ is a (two-sided) ideal, then for all $s_i, t_i \in R$, $i = 1, \dots, n$, we have $s_i a t_i \in R$. Therefore, $\sum_{i=1}^n s_i a t_i \in \langle a\rangle$, because an ideal is closed under addition. Thus, $RaR \subset \langle a\rangle$. □