Question 1. *Use induction to prove that* $f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^{n+1}}$, $n = 1, 2, 3, \ldots$

Solution. For the base of induction consider the case $n = 0$. Then the formula has the form

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{s-z}.$$

It is simply the Cauchy's integral formula for $f(z)$. The inductive step: suppose the formula is true for some nonnegative integer n. Prove that it is true for $n+1$. Differentiate under the sign of integral with respect to z:

$$\begin{aligned} f^{(n+1)}(z) &= (f^{(n)}(z))' \\ &= \left(\frac{n!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^{n+1}}\right)' \\ &= \frac{n!}{2\pi i}\int_C f(s)ds\left(\frac{-(n+1)\cdot(-1)}{(s-z)^{n+2}}\right) \\ &= \frac{(n+1)!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^{n+2}}, \end{aligned}$$

as desired. □