Question 1. *The recursive sequence is defined as* $a_1 = 9$*,* $a_2 = 6$*,* $a_{n+1} = \sqrt{a_{n-1}} + \sqrt{a_n}$ *,* $n > 2$*. Show that the sequence is bounded and strictly decreasing. Find its limit.*

Solution. Note that $a_3 = \sqrt{a_2} + \sqrt{a_1} = \sqrt{6} + \sqrt{9} = 3 + \sqrt{6} < \sqrt{9} + \sqrt{9} = 3 + 3 = 6 = a_2$ and $a_3 = 3 + \sqrt{6} > 3 + \sqrt{1} = 3 + 1 = 4$. Prove by induction that a_n is strictly decreasing and bounded below by 4. The base: $a_1 > a_2 > a_3 > 1$. Suppose that $1 < a_k < a_{k-1}$ for all $k \leq n$, where $n \geq 3$. Consider $k = n + 1$. Using the recursive formula we see that

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{a_{n-1}} + \sqrt{a_n}}{\sqrt{a_{n-2}} + \sqrt{a_{n-1}}} = \frac{1 + \sqrt{\frac{a_n}{a_{n-1}}}}{\sqrt{\frac{a_{n-2}}{a_{n-1}}} + 1}.$$

By inductive hypothesis $1 < a_n < a_{n-1}$ and $a_{n-2} > a_{n-1} > 1$, therefore,

$$\sqrt{\frac{a_n}{a_{n-1}}} < 1, \quad \sqrt{\frac{a_{n-2}}{a_{n-1}}} > 1,$$

and hence

$$\frac{1 + \sqrt{\frac{a_n}{a_{n-1}}}}{\sqrt{\frac{a_{n-2}}{a_{n-1}}} + 1} < \frac{1+1}{1+1} = 1.$$

This means that $a_{n+1} < a_n$. Furthermore,

$$a_{n+1} = \sqrt{a_{n-1}} + \sqrt{a_n} > \sqrt{4} + \sqrt{4} = 2 + 2 = 4.$$

Thus, a_n is strictly decreasing and bounded below by 4, therefore, we conclude that it has a limit $a \geq 4$. Taking the recursive formula $a_{n+1} = \sqrt{a_{n-1}} + \sqrt{a_n}$ and passing to the limit when $n \to \infty$, we get

$$a = \sqrt{a} + \sqrt{a} \Leftrightarrow a = 2\sqrt{a} \Leftrightarrow \sqrt{a} = 2 \Leftrightarrow a = 4.$$

Here we used the fact that $a > 0$.
Answer: $\lim_{n\to\infty} a_n = 4$. □