Question 1. *Prove the Distributive Laws:*

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$*;*

(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$*.*

Solution. (a) Prove that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Suppose $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$. The latter means that either $x \in B$, or $x \in C$. If $x \in B$, then $x \in A \cap B$, and if $x \in C$, then $x \in A \cap C$. Thus, either $x \in A \cap B$, or $x \in A \cap C$, i. e. $x \in (A \cap B) \cup (A \cap C)$.

Prove the converse inclusion. Take $x \in (A \cap B) \cup (A \cap C)$. So, either $x \in A \cap B$, or $x \in A \cap C$. In both cases $x \in A$. If $x \in A \cap B$, then $x \in B$, and if $x \in A \cap C$, then $x \in C$. So, either $x \in B$, or $x \in C$, i. e. $x \in B \cup C$. Thus, we proved $x \in A$ and $x \in B \cup C$, hence $x \in A \cup (B \cap C)$.

(b) Prove the inclusion $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. Then either $x \in A$, or $x \in B \cap C$. In the first case $x \in A$, which is a subset of both $A \cup B$ and $A \cup C$. So, $x \in (A \cup B) \cap (A \cup C)$. In the second case $x \in B \cap C$, which is a subset of $B \subset A \cup B$ and $C \subset A \cup C$. Thus, $x \in (A \cup B) \cap (A \cup C)$ in this case.

Now prove the converse inclusion. Choose $x \in (A \cup B) \cap (A \cup C)$. So $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then obviously $x \in A \cup (B \cap C)$, because $A \subset A \cup (B \cap C)$. Otherwise $x \in B$ and $x \in C$, i. e. $x \in B \cap C \subset A \cup (B \cap C)$.

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