Question 1. *Describe all homomorphisms $f : \mathbb{Z}_4 \to \mathbb{Z}_6$. Describe their kernels and ranges.*

Solution. Since $\mathbb{Z}_4$ is cyclic and $[1]_4$ generates $\mathbb{Z}_4$, any homomorphism on this group is fully determined by $f([1]_4)$. Note that

$$4f([1]_4) = f(4[1]_4) = f([0]_4) = [0]_6,$$

so the order of $f([1]_4)$ in $\mathbb{Z}_6$ should divide 4. The order of $[0]_6$ is 1, the order of $[1]_6$ is 6, the order of $[2]_6$ is 3, the order of $[3]_6$ is 2, the order of $[4]_6$ is 3 and the order of $[5]_6$ is 6. Thus, we have two cases: either $f([1]_4) = [0]_6$, or $f([1]_4) = [3]_6$.

If $f([1]_4) = [0]_6$, then for any $n = 0, \dots, 3$:

$$f([n]_4) = f(n[1]_4) = nf([1]_4) = n[0]_6 = [0]_6,$$

so f is the identity homomorphism. Then $\ker f = \mathbb{Z}_4$ and $\operatorname{im} f = \{[0]_6\}$.

Now assume $f([1]_4) = [3]_6$. Then for any $n \in \mathbb{Z}$:

$$f([n]_4) = n[3]_6 = [3n]_6 = \begin{cases} [3]_6, & n \text{ is odd}, \\ [0]_6, & n \text{ is even}. \end{cases} \tag{1}$$

Check the correctness. Suppose $[m]_4 = [n]_4$, i. e. $m - n = 4k$ for some $k \in \mathbb{Z}$. Then m is even if and only if n is even. So, $f([m]_4) = f([n]_4)$. Thus, the equality (1) is correct, so it defines a homomorphism $f : \mathbb{Z}_4 \to \mathbb{Z}_6$, because

$$f([m]_4 + [n]_4) = f([m+n]_4) = (m+n)[3]_6 = m[3]_6 + n[3]_6 = f([m]_4) + f([n]_4).$$

By (1) we see that $\ker f = \{[0]_4, [2]_4\} < \mathbb{Z}_4$ and $\operatorname{im} f = \{[0]_6, [3]_6\} < \mathbb{Z}_6$.
Answer:

1. the identity homomorphism, $\ker f = \mathbb{Z}_4$ and $\operatorname{im} f = \{[0]_6\}$;
2. $f([1]_4) = [3]_6$, $\ker f = \{[0]_4, [2]_4\}$ and $\operatorname{im} f = \{[0]_6, [3]_6\}$.

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