Conditions

Use Rieman sums and a limit to compute the exact area under the curve. y=x^2 + 5x on [3,10].

Solution

As we know, the area under the curve is the integral:

$$\int_3^{10} (x^2+5x)dx = \lim_{d(\pi)\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

Let's do a substitution:

$$x = t+3; dx = dt, x^2+5x = t^2+6t+9+5t+15 = t^2+11t+24$$

$$\int_3^{10} (x^2+5x)dx = \int_0^7 (t^2+11t+24)dt$$

Let's take the partition:

$$t_i = \frac{7i}{n}, t_n = 7$$

Let's points $\xi_i = t_i$. Then:

$$\lim_{d(\pi)\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

$$= \lim_{n\to\infty} \left(\left(\frac{7^2}{n}\cdot\frac{7}{n}+\cdots+\frac{7}{n}\cdot\left(\frac{7i}{n}\right)^2+\cdots+\frac{7}{n}\cdot 7^2\right)+11\left(\frac{7}{n}\cdot\frac{7}{n}+\cdots+\frac{7}{n}\cdot\frac{7i}{n}+\cdots+\frac{7}{n}\cdot 7\right)\right)$$

$$\left(\left(\frac{7}{n}\right)^2\cdot\frac{7}{n}+\cdots+\frac{7}{n}\cdot\left(\frac{7i}{n}\right)^2+\cdots+\frac{7}{n}\cdot 7^2\right) = \frac{7^3}{n^3}(1+\cdots+i^2+\cdots+n^2) =$$

$$= \frac{7^3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{7^3}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right) = \frac{7^3}{3}+\frac{7^3}{2n}+\frac{7^3}{6n^2}$$

$$11\left(\frac{7}{n}\cdot\frac{7}{n}+\cdots+\frac{7}{n}\cdot\frac{7i}{n}+\cdots+\frac{7}{n}\cdot 7\right) = \frac{11\cdot 49}{n^2}(1+\cdots+i+\cdots+n) = \cdots = \frac{11\cdot 49}{n^2}\frac{n(n+1)}{2}$$

$$\lim_{d(\pi)\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \lim_{n\to\infty}\left(\frac{7^3}{3}+\frac{7^3}{2n}+\frac{7^3}{6n^2}+\frac{11\cdot 49}{n^2}\frac{n(n+1)}{2}+24\cdot 7\right)$$

$$= \frac{7^3}{3}+\frac{11\cdot 49}{2}+24\cdot 7 = \frac{3311}{6}$$