Question 1. *Let f be a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$. Prove that either $f(x) = 0$ for every $x \in \mathbb{Z}$, or $f(x) = x$ for every $x \in \mathbb{Z}$.*

Solution. Consider $f(1)$. Since f is a homomorphism of rings, we have

$$f(1)^2 = f(1^2) = f(1).$$

Therefore, $f(1)(f(1) - 1) = 0$, i. e. either $f(1) = 0$ or $f(1) = 1$.

Suppose $f(1) = 0$. Then for any $x \in \mathbb{Z}$:

$$f(x) = f(x \cdot 1) = f(x)f(1) = f(x) \cdot 0 = 0,$$

so f is the zero homomorphism.

Now let $f(1) = 1$. Then for any $x > 0$ we have

$$f(x) = f(\underbrace{1 + \cdots + 1}_{x\, terms}) = \underbrace{f(1) + \cdots + f(1)}_{x\, terms} = \underbrace{1 + \cdots + 1}_{x\, terms} = x.$$

Furthermore, if $x < 0$, then $-x > 0$, so

$$f(x) = f(-(-x)) = -f(-x) = -(-x) = x.$$

Finally, $f(0) = 0$, because 0 is the zero of $\mathbb{Z}$. Thus, f is the identity homomorphism in this case. $\square$