Conditions

Uniform continuity:

Prove that if f and g are uniformly continuous on R and are bound, then f*g is uniformly continuous on R.

Solution

Consider:

$f: M \epsilon R \ \rightarrow R$

Function f is uniformly continued in M, if:

$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0: \forall x_1, x_2 \epsilon M \ |x_1 - x_2| < \delta \ |f(x_1) - f(x_2)| < \varepsilon$

For our case:

As functions are bounded, so

$\exists M_1, M_2: \forall x \epsilon R \ |f(x)| < M_1 \ , |g(x)| < M_2$

$\forall \varepsilon > 0 \ \exists \delta_1 = \delta_1(\varepsilon) > 0: \forall x_1, x_2 \epsilon R \ |x_1 - x_2| < \delta_1 \ |f(x_1) - f(x_2)| < \frac{\varepsilon}{2M_1}$

$\forall \varepsilon > 0 \ \exists \delta_2 = \delta_2(\varepsilon) > 0: \forall x_1, x_2 \epsilon R \ |x_1 - x_2| < \delta_2 \ |g(x_1) - g(x_2)| < \frac{\varepsilon}{2M_2}$

Fix $\varepsilon > 0$, $\exists \delta = \min(\delta_1, \delta_2)$. Consider:

$|f(x_1)g(x_1) - f(x_2)g(x_2)| = |f(x_1)g(x_1) - f(x_1)g(x_2) + f(x_1)g(x_2) - f(x_2)g(x_2)| \leq$

$\leq |f(x_1)g(x_1) - f(x_1)g(x_2)| + |f(x_1)g(x_2) - f(x_2)g(x_2)| <$

$< M_1|g(x_1) - g(x_2)| + M_2|f(x_1) - f(x_2)| \leq \varepsilon$

Q.E.D.