Assume that all barrels of wine have precisely cylindrical shape of radius r and height h. The wine merchant prices each barrel by the length d of a dipstick that can be inserted diagonally through a hole in the center of the top disc of the barrel and that reaches the edge of the bottom. For a fixed length d, determine the ratio r/h that maximizes

Solution

First we have $d^2 = r^2 + h^2$, with d fixed, and so $r^2 = d^2 - h^2$.
Now the volume of a barrel will be

$$V = \pi * r^2 * h = \pi * (d^2 - h^2) * h = \pi * ((d^2) * h - h^3).$$

Now differentiate V with respect to h and set equal to 0 to find the critical points:
$\frac{dV}{dh} = \pi * (d^2 - 3 * h^2) = 0$ when $d^2 = 3 * h^2$.
Now $\frac{d}{dh}\left(\frac{dV}{dh}\right) = -6\pi * h < 0$ for any positive h, so by the second derivative test there is a maximum for V at the critical point obtained above.
So when $d^2 = 3 * h^2$ we have

$$3 * h^2 = r^2 + h^2 \rightarrow 2 * h^2 = r^2 \rightarrow 2 = \left(\frac{r}{h}\right)^2 \rightarrow \frac{r}{h} = \sqrt{2},$$