Consider a game with two players, Jim and Annie. Annie has a red die and Jim has a white die. They roll their dice and note the number on the upper face. Annie wins if her score is higher than Jim's (note that Jim wins if the scores are the same). Investigate the game when they can roll the dice more than twice, but not necessarily the same number of times.

Solution

Annie is player A, Jim is player B.

Consider the game where player A may roll her die n times and player B may roll his die m times, with n and m being positive integers that are not necessarily distinct. In order for a specific number p to be recognized as player A's highest roll, it must be equal to or greater than all of the other rolls that player A makes. There are n ways that this could happen: player A may roll p anywhere from one to n times. Suppose that player A rolls p across c trials, where $1 \le c \le n$. Then she must roll numbers less than p across her remaining $n - c$ trials. Across all n trials, there are $\frac{n}{c}$ ways to arrange the p's that player A rolls. Each of the remaining slots may be filled with any integer between one and p; therefore there are $\binom{n}{c} \times (p-1)^{n-c}$ different ways to do this for each p out of 6^n total possible outcomes; therefore the probability that p is the highest number that player A rolls is $\sum_{c=1}^{n}(\frac{\binom{n}{c}\times(p-1)^{n-c}}{6^n})$. Similarly, in the probability that an integer q between one and six is recognized as player B's highest roll is $\sum_{d=1}^{m}\frac{\binom{m}{d}\times(q-1)^{m-d}}{6^m}$, where d is the number of times that he rolls q.

Note that, in order for player A to win by rolling p, player B must roll an integer q across d trials such that $1 \le q < p$. Thus q can be expressed as $p - f$, where $1 \le f < p$ and f is an integer. The probability that a specific p will allow player A to win is therefore $\sum_{c=1}^{n}\left(\frac{\binom{n}{c}\times(p-1)^{n-c}}{6^n}\right) \times \left(\sum_{f=1}^{p-1}(\sum_{d=1}^{m}\frac{\binom{m}{d}\times((p-f)-1)^{m-d}}{6^m})\right)$; thus the probability that player A will win is $P(A\ Wins) =$ $\sum_{p-1}^{6}\left(\sum_{c=1}^{n}\left(\frac{\binom{n}{c}\times(p-1)^{n-c}}{6^n}\right) \times \left(\sum_{f=1}^{p-1}(\sum_{d=1}^{m}\frac{\binom{m}{d}\times((p-f)-1)^{m-d}}{6^m})\right)\right)$; note that this equation will only yield correct probabilities in a game consisting two people playing with fair, six-sided dice.

To verify this equation, some game variations have been analyzed first by counting the number of outcomes in which player A wins to determine the probability of player A winning in those scenarios. Then, the probabilities of player A winning in each of the game types is calculated using the above formula with the help of script on Microsoft Excel

Game Variation	"Counted" Probability	Calculated Probability (Equation)
A has one roll, B has one roll	$\frac{5}{12}$	$\frac{5}{12}$
A has two rolls, B has one roll	$\frac{125}{216}$	$\frac{125}{216}$
A has two rolls, B has two rolls	$\frac{317}{810}$	$\frac{317}{810}$
A has three rolls, B has one roll	$\frac{95}{144}$	$\frac{95}{144}$