Conditions

Suppose p is a polynomial with n distinct real roots. Show that p' has at least n−1 distinct real roots.

Solution

This can be very easy proved by using Rolle's Theorem.

In calculus, **Rolle's theorem** essentially states that a differentiable function which attains equal values at two distinct points must have a point somewhere between them where the first derivative (the slope of the tangent line to the graph of the function) is zero.

The polynomial is a differentiable function, and as we have n roots, so we have n-1 intervals, where at two distinct points we have equal values (zero, as they are roots).

For example, let's our roots are (ranged from smallest value to largest):

$x_1, x_2, \dots, x_n$

Then we have following intervals:

$(x_1, x_2), (x_2, x_3), \dots (x_{n-1}, x_n)$

As we see – the amount of intervals is n-1.

And:

$P(x_1) = P(x_2) = \cdots = P(x_n) = 0$

As they all are roots of polynomial P(x).

So, in each interval let's use the Rolle's Theorem, and we'd got a proof.

This Theorem guarantee us one zero of derivative function, but there is nothing said about "one and only one", so we can claim that $P'(x)$"at least n-1 roots".

Q.E.D.