Conditions

Suppose that {v1.....Vn} is a basis for the vector space V. Given any vector v element of V, we can express v as a linear combination v = X1V1+.....+XnVn. The uniqueness of this expression means that mapping v to the n-tuple of coefficients (X1...Xn) defines a function O: V - R^n. Prove that this function is a bijection.

Solution

A bijection (or bijective function or one-to-one correspondence) is a function giving an *exact* pairing of the elements of two sets. Every element of one set is paired with exactly one element of the other set, and every element of the other set is paired with exactly one element of the first set. There are no unpaired elements. In formal mathematical terms, a bijective function *f*: *X* → *Y* is a one to one and onto mapping of a set *X* to a set *Y*.

A bijection from the set *X* to the set *Y* has an inverse function from *Y* to *X*. If *X* and *Y* are finite sets, then the existence of a bijection means they have the same number of elements. For infinite sets the picture is more complex, leading to the concept of cardinal number, a way to distinguish the various sizes of infinite sets.

The definition of bijection is below.

Function *f*: *X* → *Y* is called bijection (and marked *f*: *X* <-> *Y*) if :

1. $\forall x_1 \epsilon X, \forall x_2 \epsilon X, (f(x_1) = f(x_2) \ \rightarrow \ x_1 = x_2)$

2. $\forall y \epsilon Y, \exists x \epsilon X \ f(x) = y$

Let's check these conditions for our function O.

$O: V \ \rightarrow R^n, O(v) = (x_1, \dots, x_n)$

$\forall \, u, w \, \epsilon \, V$ let's consider if $(O(u) = O(w)$ involves $u = w)$

$O(u): u = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \ ; O(u) = (u_1, \dots, u_n)$

$O(w): w = w_1 v_1 + w_2 v_2 + \cdots + w_n v_n \ ; O(w) = (w_1, \dots, w_n)$

If $O(u) = O(w)$, then $(u_1, \dots, u_n) = (w_1, \dots, w_n)$ and this could be if and only if $u_i = w_i \ \forall i = \overline{1, n}$.

That means, the 1st condition is true for function O.

Let's check the 2nd.

As we know

$\forall y = (y_1, \dots, y_n) \ \epsilon \ O(V) \ \exists$ linear combination $y_1 v_1 + y_2 v_2 + \cdots + y_n v_n$,and as we know that $y \ \epsilon \ V$, so it means that $\exists$ some vector a, for which $a = y_1 v_1 + y_2 v_2 + \cdots + y_n v_n$. If it was wrong, then the element y couldn't exist in O(V).

So, the second condition is also true.

The function O is a bijection on vector space V.