Conditions

for non-trivial subspaces U and W of a finite-dimensional vector space V, define
U+W := {u+w | u element of U and w element of W}. Prove that U+W is a subspace of V

Solution

Consider a finite-dimensional vector space V with a field P.

U,W – subspaces of V

If we want prove, that U+W is a subspace of V, we must use the Subspace criterion. It claims, that:
Non-trivial set X∈ V is a subspace of V if and only if, when:

1. $\forall x, y \in X, x + y \in X$
2. $\forall a \in P, \forall x \in X, ax \in X$

First we must prove, that U+W is in V.

Let's V is a n-dimensional space. Fix a random basis $(v_1, v_2, \dots v_n) \in V$

As U is a subset of V, then:

$\forall u \in U, u = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n, a_1, \dots, a_n \in P$

As W is a subset of V, then:

$\forall w \in W, w = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n, b_1, \dots, b_n \in P$

$$u + v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + b_1 v_1 + b_2 v_2 + \cdots + b_n v_n, b_1, \dots, b_n$$
$$= (a_1 + b_1) v_1 + (a_2 + b_2) v_2 + \cdots + (a_n + b_n) v_n$$

So, u+w is an element of V for each u and w, so U+W is in V.

The first condition of the Subspace Criterion is obvious:

$\forall u_1, u_2 \in U, u_1 + u_2 \in U$ (as U is a subspace of V and the criterion for U is correct)

$\forall w_1, w_2 \in W, w_1 + w_2 \in W$ (as W is a subspace of V and the criterion for W is correct)

$\forall u_1, u_2 \in U,\ \forall w_1, w_2 \in W,\ \ u_1 + w_1 + u_2 + w_2$ is in U+W, because $u_1 + u_2 \in U$ and $w_1 + w_2 \in W$.

The second condition of the Subspace Criterion is obvious too:

As U and W are subspaces, then the 2nd condition is correct for them:

$\forall a \in P, \forall u \in U, au \in U$

$\forall b \in P, \forall w \in W, bw \in W$

For U+W:

$\forall c \in P, q \in (U + W)$

Consider c(U+W):

$c(U + W) = cu + cw, u \in U, w \in W$

As we know,

$\forall a \in P, \forall u \in U, au \in U,$ especially, for a=c

$\forall b \in P, \forall w \in W, bw \in W$, especially, for b=c

As $cu \in U$, $cw \in W$, than $cu + cw \in U + W$ (by the definition of U+W)

Then the 2nd condition is correct.

Q.E.D.