Conditions

look at the matrix
A =

1	2	-2	-1
0	1	2	-3
1	4	2	-7
1	3	0	-4

what is the dimension of the row space of the matrix A ?
what is the dimension of the column space of the matrix A ?
what is the dimension of the null space of the matrix A ?
Give the null space for matrix A ?

Solution

$$A = \begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & 2 & -3 \\ 1 & 4 & 2 & -7 \\ 1 & 3 & 0 & -4 \end{pmatrix}$$

The dimension of the row space of a matrix is called row rank.

The **row rank** of a matrix *A* is the maximum number of linearly independent row vectors of *A*. Equivalently, the column rank of *A* is the dimension of the column space of *A*, while the row rank of *A* is the dimension of the row space of *A*.

A result of fundamental importance in linear algebra is that the **column rank** and the **row rank** are always equal. This number (i.e. the number of linearly independent rows or columns) is simply called the **rank** of *A*.

It's obvious to notice, that the 4^{th} row is equal to a 3^{rd} minus 2^{nd} . Let's do this linear transformation:

$$\begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & 2 & -3 \\ 1-0 & 4-1 & 2-2 & -7-(-3) \\ 1 & 3 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & 2 & -3 \\ 1 & 3 & 0 & -4 \\ 1 & 3 & 0 & -4 \end{pmatrix}$$

We have 3 rows left

$$\begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & 2 & -3 \\ 1 & 3 & 0 & -4 \end{pmatrix}$$

Let's check, if other 3 rows are linear independent:

$$\begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & 2 & -3 \\ 1 & 3 & 0 & -4 \end{pmatrix} \sim (3rd\ minus\ 1st) \sim \begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & 2 & -3 \\ 0 & 1 & 2 & -3 \end{pmatrix}$$

As we see, the 2^{nd} row is the linear combination of 3^{rd} minus 1^{st}.

So we have at least 2 rows, which were linear dependent.

$$\begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & 2 & -3 \end{pmatrix}$$

As we see, the 1st element in 2nd row is 0, and the 1st element in 1st row is 1. This elements couldn't be transform to each other by any linear transformation, so that means, that the rank of A is 2

<u>The row space of A is equal to a column space of A and is equal to 2.</u>

The null space of matrix *A* is the set of all vectors **x** for which *A***x** = **0**. The product of the matrix *A* and the vector **x** can be written in terms of the dot product of vectors:

$$A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix},$$

where $\mathbf{r}_1, \ldots, \mathbf{r}_m$ are the row vectors of *A*. Thus *A***x** = **0** if and only if **x** is orthogonal (perpendicular) to each of the row vectors of *A*.

The matrix equation **Ax** = **0** is equivalent to a homogeneous system of linear equations:

$$\mathbf{Ax} = \mathbf{0} \quad \Leftrightarrow \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \;\; &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

It's obvious that our system has a form:

$$\begin{cases} x_1 + 2x_2 - 2x_3 - x_4 = 0 \\ x_2 + 2x_3 - 3x_4 = 0 \end{cases}$$

The dimension of null space is 2, because we have 2 free vectors, which is from R, and only 2 could be solved for fixed values of these 2. Really, let's $x_3 \epsilon R, x_4 \epsilon R$

Then

$$\begin{cases} x_1 = -2x_2 + 2x_3 + x_4 \\ x_2 = -2x_3 + 3x_4 \end{cases}$$

For $x_3 = 1, x_4 = 0$:

$x_2 = -2$

$x_1 = 4 + 2 = 6$

For $x_3 = 0, x_4 = 1$:

$x_2 = 3$

$x_1 = -6 + 1 = -5$

The basis of null space is:

$(\mathbf{-5}, \mathbf{3}, \mathbf{0}, \mathbf{1}), (\mathbf{6}, \mathbf{-2}, \mathbf{1}, \mathbf{0})$