Conditions

Using the epsilon-delta definition of the limit, prove that if lim x-->a f(x) and lim x-->a g(x) exist, then lim x-->a [f(x) + g(x)] = lim x-->a f(x) + lim x-->a g(x).

Solution

Definition. The limit of function f(x) is equal to F (when x →a), if:

$\forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon)\ \forall x\colon |x - a| < \delta\ |f(x) - F| < \varepsilon$

Let's write, what means that f(x) and g(x) have limits when x→a:

$\forall \varepsilon > 0\ \exists \delta_1 = \delta_1(\varepsilon)\ \forall x\colon |x - a| < \delta_1\ |f(x) - F| < \varepsilon$

$\forall \varepsilon > 0\ \exists \delta_2 = \delta_2(\varepsilon)\ \forall x\colon |x - a| < \delta_2\ |g(x) - G| < \varepsilon$

Fix $\varepsilon > 0$, consider $\delta = \max(\delta_1; \delta_2)$

$|f(x) - F + g(x) - G| \leq |f(x) + g(x) - (F + G)| \leq |f(x) - F| + |g(x) - G| < 2\varepsilon$

As we can see:

$\forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon) = \max(\delta_1; \delta_2)\ \forall x\colon |x - a| < \delta\ \ |f(x) + g(x) - (F + G)| < 2\varepsilon$

This means, that $f(x) + g(x)$ has a limit and it exactly equal to the sum of F and G.

Q.E.D.