Conditions

Indefinite limit problem:

Show that If an < 0 and lim an = 0, then lim 1/an = negative infinity.

Solution

Let's show that $lim\frac{1}{a_n} = -\infty$ by using a definition:

$limx_n = -\infty \quad \leftrightarrow \quad \forall\varepsilon > 0 \; \exists N = N(\varepsilon) \;\; \forall n \geq N \; x_n < -\varepsilon$

Fix $\varepsilon > 0$

Let's find $N = N(\varepsilon): \forall n \geq N \; \frac{1}{a_n} < -\varepsilon$

As we know, $lima_n = 0$

$\forall\varepsilon > 0 \; \exists N' = N'(\varepsilon) \;\; \forall n \geq N' \;\; |a_n| < \varepsilon$

As we know, $a_n < 0 \; \forall n\epsilon N$. So, we can remove the modulo in the next way:

$\forall\varepsilon > 0 \; \exists N' = N'(\varepsilon) \;\; \forall n \geq N' - a_n < \varepsilon$

$a_n > -\varepsilon,$

$$\frac{1}{a_n} < \frac{1}{-\varepsilon} = -\varepsilon'$$

As this is true $\forall\varepsilon > 0$, so $\varepsilon' - \forall$. Here $N = N + 1$ (bigger that N' for at least 1).