Question 1. *Show that, if R has an identity 1, the map $\phi : (R, \circ) \to (R, \cdot)$ sending a to $1 - a$ is a monoid isomorphism. In this case, an element a is left (right) quasi-regular iff $1 - a$ has a left (resp. right) inverse with respect to ring multiplication.*

Solution. Indeed,

$$\phi(a \circ b) = 1 - a \circ b = 1 - a - b + ab = (1 - a)(1 - b) = \phi(a)\phi(b),$$

so, ϕ is a homomorphism of semigroups. Moreover, $\phi(0) = 1$, so ϕ is a homomorphism of monoids (0 is the identity of $(R, \circ)$). The bijectivity of ϕ is obvious: $1-a = 1-b \Leftrightarrow a = b$, $a = 1-(1-a)$. Thus, ϕ is an isomorphism. Note that

$$a \circ b = 0 \Leftrightarrow 1 = \phi(a \circ b) = \phi(a)\phi(b) = (1 - a)(1 - b).$$

So, a is left (right) quasi-regular iff $1 - a$ is left (resp. right) invertible. $\square$

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