

We specialize following result for case of finite cardinal, and obtain our statement.

For any infinite cardinal $\beta \leq \alpha$, let $E_\beta = \{f \in E : \text{rank}(f) < \beta\}$, where $\text{rank}(f)$ denotes the cardinal number $\dim_D f(V)$. Since $\text{rank}(g'fg) \leq \text{rank}(f)$, E_β is an ideal of E . We claim that the ideals of E are (0) , E and the E_β 's. For this, we need the following crucial fact. (*) *If $f, h \in E$ are such that $\text{rank}(h) \leq \text{rank}(f)$, then $h \in EfE$.*

Indeed, write $V = \ker(h) \oplus V_1 = \ker(f) \oplus V_2$. Fix a basis $\{u_i : i \in I\}$ for V_1 , and a basis $\{v_j : j \in J\}$ for V_2 . We have $|I| = \text{rank}(h) \leq \text{rank}(f) = |J|$, so let us assume, for convenience, that $I \subseteq J$. Define $g \in E$ such that $g(\ker(h)) = 0$, and $g(u_i) = v_i$ for all $i \in I$. Noting that $\{f(v_j) : j \in J\}$ are linearly independent, we can also find $g' \in E$ such that $g'(f(v_j)) = h(u_j)$ for all $j \in I$. Then $h = g'fg$. In fact, both sides are zero on $\ker(h)$, and on u_i ($i \in I$) we have $g'fg(u_i) = g'f(v_i) = h(u_i)$. This proves (*). Now consider any ideal $A \neq 0, E$. Then, for any $f \in A$, $\text{rank}(f) < \alpha$. For, if $\text{rank}(f) = \alpha = \text{rank}(\text{Id})$, then (*) implies $\text{Id} \in EfE \subseteq A$, a contradiction. Since the class of cardinal numbers is well-ordered, there exists a cardinal $\beta \leq \alpha$ which is least among cardinals larger than $\text{rank}(f)$ for every $f \in A$. We leave to the reader the easy task of verifying that β is an infinite cardinal. Clearly, $A \subseteq E_\beta$ by the definition of E_β . We finish by showing that $E_\beta \subseteq A$. Let $h \in E_\beta$, so $\text{rank}(h) < \beta$. By the choice of β , we must have $\text{rank}(h) \leq \text{rank}(f)$ for some $f \in A$. But then, by (*), $h \in EfE \subseteq A$, as desired. Thus, ring E has exactly three ideals: 0 , E , and the ideal consisting of endomorphisms of finite rank.