We look for a counterexample *R*, which, by (2), must be neither commutative nor reduced. An obvious choice is a matrix ring *R* = M2(*S*), where *S* is a nonzero ring. To see this, let *f* = *a* + *bx* and *g* = *c* + *dx*, where

$$a = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ c = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},\ d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

We have *ac* = *bd* = 0 $\in R$ and also *ad* + *bc* = $\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0 \ \in R,$

so *fg* = 0 $\in A$. However, *for any r* $\in R$, *f* · *r* = 0 *implies r* = 0. To see this, let

$$r = \begin{pmatrix} p & q \\ s & t \end{pmatrix}.$$

From *f* · *r* =0, we have 0 = *ar* = $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p & q \\ s & t \end{pmatrix}$, and 0 = *br* = $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p & q \\ s & t \end{pmatrix}$

which forces *p* = *q* = 0 = *s* = *t*, so *r* = 0 $\in R$.