Answer on Question #16803 – Math – Discrete Mathematics

Question

Prove that

$A\backslash(A\backslash B) = A \cap B$

Solution

It is known that

$$A\backslash B = A \cap B', \quad \textbf{(1)}$$

where $B' = \{x | x \notin B\}$.

$$B \cap B' = \emptyset, \quad \textbf{(2)}$$

$$B'' = (B')' = B \quad \textbf{(3)}$$

$$(C \cap D)' = C' \cup D' \text{ (De Morgan's Law)} \quad \textbf{(4)}$$

$$C \cap (D \cup E) = (C \cap D) \cup (C \cap E) \text{ (Distributive Law)} \quad \textbf{(5)}$$

$$\emptyset \cup C = C \quad \textbf{(6)}$$

Consider

$$(A\backslash B)' = (A \cap B')' = A' \cup B'' = A' \cup B \quad \textbf{(7)}$$

Method 1

$A\backslash(A\backslash B) = |(1)| = A \cap (A\backslash B)' = |(1)| = A \cap (A \cap B')' = |(4)| = A \cap (A' \cup B'') = |(5)| =$

$= (A \cap A') \cup (A \cap B'') = |(2)| = \emptyset \cup (A \cap B'') = |(3)| = \emptyset \cup (A \cap B) = |(6)| =$

$= A \cap B.$

All transformations are equivalent due to correctness of laws (1) to (6).

This proves that $A\backslash(A\backslash B) = A \cap B$.

Method 2

If $x \in A\backslash(A\backslash B)$, then it means that

$$(x \in A) \quad \textbf{(8)}$$

and

$$(x \notin A\backslash B). \quad \textbf{(9)}$$

If (9) holds true, then $x \in (A\backslash B)'$, hence using (7) obtain that

$$x \in A' \quad \textbf{(10)}$$

or

$$x \in B \quad \textbf{(11)}$$

Case (10) contradicts to (8), therefore, case $x \in A'$ is not possible. Case $x \in B$ is possible.

We know $x \in A$ according to (8) and $x \in B$ according to (11). Hence $(x \in A)$ and $(x \in B)$, therefore, $x \in A \cap B$.

So we showed that

$$A \backslash (A \backslash B) \subset A \cap B. \qquad \textbf{(12)}$$

This means $A \backslash (A \backslash B)$ is a subset of $A \cap B$.

On the other hand, if $x \in A \cap B$, then $(x \in A)$ and $(x \in B)$, so $x \in A$, but $x \notin A \backslash B$, because $x \in B$.

So $x \in A \backslash (A \backslash B)$. Then

$$A \cap B \subset A \backslash (A \backslash B), \qquad \textbf{(13)}$$

This means $A \cap B$ is a subset of $A \backslash (A \backslash B)$.

Both (12) and (13) are true, hence these sets are equal:

$$A \cap B = A \backslash (A \backslash B)$$