Changing notations, we write here $\overline{A} = A/A \cdot (x4 + 1)$. Define $\varphi : A \to M(2,C)$ by

$\varphi(x) = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$ and $\varphi(a) = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix}$ for $a \in C$.

Since

$$\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix} = \begin{pmatrix} 0 & i\sigma(a) \\ a & 0 \end{pmatrix} = \begin{pmatrix} \sigma(a) & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$$

φ gives a ring homomorphism from A to M(2,C). Again, φ induces a ring homomorphism $\varphi : A \to M(2,C)$, since

$$\varphi(x^4 + 1) = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}^4 + I = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}^2 + I = 0$$

By a straightforward computation, for $b_k, c_k \in R$:

$$\overline{\phi}\left(\sum_{k=0}^{3} \overline{(b_k + ic_k)x^k}\right) = \begin{pmatrix} b_0 + c_0 i + i(b_2 + c_2 i) & -(b_3 + ic_3) + i(b_1 + c_1 i) \\ (b_3 - ic_3) + i(b_1 - c_1 i) & b_0 - c_0 i + i(b_2 - c_2 i) \end{pmatrix}$$

Clearly, this is the zero matrix only if all $b_k, c_k = 0$. Therefore, φ is one-one. Since $\overline{\phi}$ is an R-homomorphism and both A and M(2,C) have dimension 8 over R, it follows that $\overline{\phi}$ is an isomorphism.