The proof consist of several parts which we will give for completeness. Let K denote kerf . The following calculation validates that for every $g \in G$ and $k \in K$:

$$\begin{aligned} f(gkg^{-1}) &= f(g)\, f(k)\, f(g)^{-1} && (f \text{ is an homomorphism}) \\ &= f(g)\, 1_H\, f(g)^{-1} && (\text{definition of } K) \\ &= 1_H \end{aligned}$$

Hence, gkg^{-1} is in K . Therefore, K is a normal subgroup of G and G/K is well-defined.

To prove the theorem we will define a map from G/K to the image of f and show that it is a function, a homomorphism and finally an isomorphism.

Let $\theta: G/K \longrightarrow \mathrm{Im} f$ be a map that sends the coset gK to f(g) .

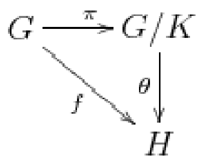
Since θ is defined on representatives we need to show that it is well defined. So, let g_1 and g_2 be two elements of G that belong to the same coset (i.e. $g_1K=g_2K$). Then, $g_1^{-1}g_2$ is an element of K and therefore $f(g_1^{-1}g_2)=1$ (because K is the kernel of G). Now, the rules of homomorphism show that $f(g_1)^{-1}f(g_2)=1$ and that is equivalent to $f(g_1)=f(g_2)$ which implies the equality $\theta(g_1K)=\theta(g_2K)$.

Next we verify that θ is a homomorphism. Take two cosets g_1K and g_2K , then:

$$\begin{aligned} \theta(g_1K \cdot g_2K) &= \theta(g_1g_2K) && (\text{operation in } G/K) \\ &= f(g_1g_2) && (\text{definition of } \theta) \\ &= f(g_1)f(g_2) && (f \text{ is an homomorphism}) \\ &= \theta(g_1K)\theta(g_2K) && (\text{definition of } \theta) \end{aligned}$$

Finally, we show that θ is an isomorphism (i.e. a bijection). The kernel of θ consists of all cosets gK in G/K such that f(g)=1 but these are exactly the elements g that belong to K so only the coset K is in the kernel of θ which implies that θ is an injection. Let h be an element of Imf and g its pre-image. Then, θ(gK) equals f(g) thus θ(gK)=h and therefore θ is surjective.

The theorem is proved. Some version of the theorem also states that the following diagram is commutative:



were π is the natural projection that takes $g \in G$ to gK . We will conclude by verifying this. Take g in G then, $\theta(\pi(g))=\theta(gK)=f(g)$ as needed.