Question 1. *Let $\alpha = (a_1, a_2, a_3, \dots, a_L)$ be a cycle of length L. Prove that α^2 is a cycle if and only if L is odd.*

Solution. Suppose $L = 2n$, $n \in \mathbb{N}$. Then

$$a_1 \stackrel{\alpha^2}{\to} a_3 \stackrel{\alpha^2}{\to} a_5 \stackrel{\alpha^2}{\to} \dots \stackrel{\alpha^2}{\to} a_{2n-3} \stackrel{\alpha^2}{\to} a_{2n-1} \stackrel{\alpha^2}{\to} a_1 \stackrel{\alpha^2}{\to} \dots$$

$$a_2 \stackrel{\alpha^2}{\to} a_4 \stackrel{\alpha^2}{\to} a_6 \stackrel{\alpha^2}{\to} \dots \stackrel{\alpha^2}{\to} a_{2n-2} \stackrel{\alpha^2}{\to} a_{2n} \stackrel{\alpha^2}{\to} a_2 \stackrel{\alpha^2}{\to} \dots$$

Therefore, α^2 is the product of two independent cycles:

$$\alpha^2 = (a_1, a_3, \dots, a_{2n-1})(a_2, a_4, \dots, a_{2n}).$$

Now suppose $L = 2n - 1$, $n \in \mathbb{N}$. Then

$$a_1 \stackrel{\alpha^2}{\to} a_3 \stackrel{\alpha^2}{\to} \dots \stackrel{\alpha^2}{\to} a_{2n-1} \stackrel{\alpha^2}{\to} a_2 \stackrel{\alpha^2}{\to} a_4 \stackrel{\alpha^2}{\to} \dots \stackrel{\alpha^2}{\to} a_{2n-2} \stackrel{\alpha^2}{\to} a_1 \stackrel{\alpha^2}{\to} \dots$$

So, α^2 is the cycle

$$\alpha^2 = (a_1, a_3, \dots, a_{2n-1}, a_2, a_4, \dots, a_{2n-2}).$$

□