Question 1. *Let $C(\mathbb{R})$ denote the ring of all continuous real-valued functions on $\mathbb{R}$, with the operations of pointwise addition and pointwise multiplication. Which of the following form an ideal in this ring?*

(a) *The set of all C^∞-functions with compact support.*

(b) *The set of all continuous functions with compact support.*

(c) *The set of all continuous functions which vanish at infinity, i. e. functions f such that $\lim_{x\to\infty} f(x) = 0$*

Solution. Obviously, all these three subsets of $C(\mathbb{R})$ are subgroups of the additive group of $C(\mathbb{R})$. Consider the result of multiplication of any element of each subset by an element of $C(\mathbb{R})$ and verify whether it is again an element of the subset.

(a) Consider $f \in C^\infty(\mathbb{R})$ with compact support, such that $f(0) \neq 0$, and $g(x) = |x|$. It is obvious, that $g \in C(\mathbb{R})$. Prove that $fg \notin C^\infty(\mathbb{R})$. Indeed, the right derivative of fg at $x = 0$ is

$$(f(x)x)'|_{x=0} = (f'(x)x + f(x))|_{x=0} = f(0),$$

while the left derivative of fg at $x = 0$ is

$$(-f(x)x)'|_{x=0} = (-f'(x)x - f(x))|_{x=0} = -f(0).$$

Since $f(0) \neq 0$, the right derivative does not coincide with the left one. So, fg is not differentiable at $x = 0$ and thus $fg \notin C^\infty(\mathbb{R})$. This shows that the set of C^∞-functions with compact support is not an ideal in $C(\mathbb{R})$.

(b) Let $f \in C(\mathbb{R})$, $\mathit{supp}\, f = \overline{A}$, where $A = \{x \in \mathbb{R} \mid f(x) \neq 0\}$, $g \in C(\mathbb{R})$, $\mathit{supp}\, g = \overline{B}$, where $B = \{x \in \mathbb{R} \mid g(x) \neq 0\}$. Obviously, $fg \in C(\mathbb{R})$. Furthermore, $\mathit{supp}\, fg = \overline{A \cap B} \subset \overline{A} \cap \overline{B} = \mathit{supp}\, f \cap \mathit{supp}\, g$. So, if $\mathit{supp}\, f$ is compact, then $\mathit{supp}\, fg$ is a closed subset of compact and therefore it is also compact. Thus, the set of all continuous functions with compact support is an ideal in $C(\mathbb{R})$.

(c) Consider $f(x) = e^{-x}$ and $g(x) = e^{x}$. Note that $f, g \in C(\mathbb{R})$ and f vanishes at infinity. But $f(x)g(x) = 1$, so fg does not vanish at infinity. Thus, this set is not an ideal in $C(\mathbb{R})$.

Answer:

(a) it is not an ideal;

(b) it is an ideal;

(c) it is not an ideal.

□