$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & 3 & 5 \end{pmatrix}, b_0 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$Ax = b_0$

$Ax = b_1$

First system has infinitely many solutions, second has no solutions. (rank(A|b_0)=2=rank(A), and rank(A|b_1)=3 not equal to rank(A)).