

$$3xy'' + (1-x)y' - y = 0$$

Since the differential equation has non-constant coefficients, we cannot assume that a solution is in the form $y = e^{\alpha x}$. Instead, we use the fact that the second order linear differential equation must have a unique solution. We can express this unique solution as a power series

$$y = \sum_{n=0}^{\infty} a_n x^n$$

If we can determine the a_n for all n , then we know the solution. Fortunately, we can easily take derivatives:

Notice that 0 is a singular point of this differential equation. We will not be able to find a solution in the form $\sum a_n y^n$, since the solution will not be differentiable at zero. Alternatively, we find a solution in the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

This is the power series centered about $x=1$, which is not a singular point. Now take derivatives

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$3x \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + (1-x) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

We would like to combine like terms, but there are two problems. The first is the powers of x do not match and the second is that the summations begin in differently. We will first deal with the powers of x . We shift the index of the first, second and last summation by letting

$$3(x-1+1) \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - (x-1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$3(x-1) \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + 3 \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$- (x-1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$3 \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} + 3 \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=1}^{\infty} n a_n (x-1)^n$$

$$- \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$3 \sum_{n=1}^{\infty} n(n+1)a_{n+1}(x-1)^n + 3 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

Some summation begins at 1 while the first and second begin at 0. We deal with this by pulling out the 1th term.

$$3 \sum_{n=1}^{\infty} n(n+1)a_{n+1}(x-1)^n + 3 * 2a_2 + 3 \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^n - a_0 - \sum_{n=1}^{\infty} a_n(x-1)^n = 0$$

$$6a_2 - a_0 + \sum_{n=1}^{\infty} [3n(n+1)a_{n+1} + 3(n+2)(n+1)a_{n+2} - na_n - a_n](x-1)^n = 0$$

$$a_2 = \frac{1}{6}a_0$$

$$a_{n+2} = \frac{a_n - 3na_{n+1}}{3(n+2)}$$

We need two pair linearly independent solution, so assume

$$\text{Assume } a_0 = 1, \quad a_1 = 0, \quad \text{then } a_2 = \frac{1}{6}, \quad a_3 = -\frac{1}{18}$$

$$y_1 = 1 + \frac{1}{6}x^2 - \frac{1}{18}x^3 + \dots$$

$$\text{Assume } a_0 = 1, \quad a_1 = 1 \quad \text{then } a_2 = \frac{1}{6} \quad a_3 = \frac{1}{18}$$

$$y_2 = 1 + x + \frac{1}{6}x^2 + \frac{1}{18}x^3 + \dots$$

General solution is

$$y = C_1y_1 + C_2y_2$$