$$3xy'' + (1-x)y' - y = 0$$

Since the differential equation has non-constant coefficients, we cannot assume that a solution is in the form $y = e^{ax}$. Instead, we use the fact that the second order linear differential equation must have a unique solution. We can express this unique solution as a power series

$$y = \sum_{n=0}^{\infty} a_n x^n$$

If we can determine the $a_n$for all n, then we know the solution. Fortunately, we can easily take derivatives:

Notice that 0 is a singular point of this differential equation. We will not be able to find a solution in the form $\sum a_n y^n$, since the solution will not be differentiable at zero. Alternatively, we find a solution in the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

This is the power series centered about x=1, which is not a singular point. Now take derivatives

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$3x \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + (1-x) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

We would like to combine like terms, but there are two problems. The first is the powers of x do not match and the second is that the summations begin in differently. We will first deal with the powers of x. We shift the index of the first, second and last summation by letting

$$3(x-1+1) \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - (x-1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$3(x-1) \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + 3 \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$- (x-1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$3 \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} + 3 \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=1}^{\infty} n a_n (x-1)^n$$

$$- \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$3\sum_{n=1}^{\infty} n(n+1)a_{n+1}(x-1)^n + 3\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

Some summation begins at1while the first and second begin at 0. We deal with this by pulling out the 1^{th} term.

$$3\sum_{n=1}^{\infty} n(n+1)a_{n+1}(x-1)^n + 3*2a_2 + 3\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^n - a_0 - \sum_{n=1}^{\infty} a_n(x-1)^n = 0$$

$$6a_2 - a_0 + \sum_{n=1}^{\infty}[3n(n+1)a_{n+1} + 3(n+2)(n+1)a_{n+2} - na_n - a_n](x-1)^n = 0$$

$$a_2 = \frac{1}{6}a_0$$

$$a_{n+2} = \frac{a_n - 3na_{n+1}}{3(n+2)}$$

We need two pair linearly independent solution, so assume

Assume $a_0 = 1,\quad a_1 = 0,\quad$ then $\quad a_2 = \frac{1}{6},\quad a_3 = -\frac{1}{18}$

$$y_1 = 1 + \frac{1}{6}x^2 - \frac{1}{18}x^3 + \cdots$$

Assume $a_0 = 1,\quad a_1 = 1\quad$ then $a_2 = \frac{1}{6}\quad a_3 = \frac{1}{18}$

$$y_2 = 1 + x + \frac{1}{6}x^2 + \frac{1}{18}x^3 + \cdots$$

General solution is

$$y = C_1y_1 + C_2y_2$$