

Is there a connection between the zeta function and prime numbers?

If there is, then show it, else explain why.

The connection between the zeta function and prime numbers was discovered by Leonhard Euler, who proved the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

where, by definition, the left hand side is  $\zeta(s)$  and the infinite product on the right hand side extends over all prime numbers  $p$  (such expressions are called Euler products):

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \frac{1}{1 - 2^{-s}} \cdot \frac{1}{1 - 3^{-s}} \cdot \frac{1}{1 - 5^{-s}} \cdot \frac{1}{1 - 7^{-s}} \cdot \frac{1}{1 - 11^{-s}} \cdots \frac{1}{1 - p^{-s}} \cdots$$

Both sides of the Euler product formula converge for  $\text{Re}(s) > 1$ . The proof of Euler's identity uses only the formula for the geometric series and the fundamental theorem of arithmetic. Since the harmonic series, obtained when  $s = 1$ , diverges, Euler's formula (which becomes ) implies that there are infinitely many primes.

The Euler product formula can be used to calculate the asymptotic probability that  $s$  randomly selected integers are set-wise coprime. Intuitively, the probability that any single number is divisible by a prime (or any integer),  $p$  is  $1/p$ . Hence the probability that  $s$  numbers are all divisible by this prime is  $1/p^s$ , and the probability that at least one of them is not is  $1 - 1/p^s$ . Now, for distinct primes, these divisibility events are mutually independent because the candidate divisors are coprime (a number is divisible by coprime divisors  $n$  and  $m$  if and only if it is divisible by  $nm$ , an event which occurs with probability  $1/(nm)$ .) Thus the asymptotic probability that  $s$  numbers are coprime is given by a product over all primes,

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \left(\prod_p \frac{1}{1 - p^{-s}}\right)^{-1} = \frac{1}{\zeta(s)}.$$