Question 1. *Prove that abelian group of order pq (p, q are distinct primes) is cyclic.*

Solution. Let G be an abelian group of order pq. By the fundamental theorem of finite abelian groups we have two cases: either $G \cong \mathbb{Z}_{pq}$ (the cyclic group of order pq), or $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$ (the direct sum of cyclic groups of orders p and q). Prove that $Z_p \oplus \mathbb{Z}_q \cong \mathbb{Z}_{pq}$. Indeed, let a generate $\mathbb{Z}_p$, i. e. $pa = 0$ in $\mathbb{Z}_p$, and b generate $\mathbb{Z}_q$, i. e. $qb = 0$ in $\mathbb{Z}_q$. Then consider the pair $(a, b) \in \mathbb{Z}_p \oplus \mathbb{Z}_q$. Note that

$$pq(a, b) = (pqa, pqb) = (q(pa), p(qb)) = (q \cdot 0, p \cdot 0) = (0, 0),$$

therefore, the order of (a, b) divides pq. Since $(a, b) \neq (0, 0)$, then the order cannot be equal to 1. It cannot be also equal to p or q, because $qa \neq 0$ and $pb \neq 0$ (for example, $qa = 0$ would imply that p divides q, which is impossible, since p and q are distinct primes). Therefore, the order of (a, b) is pq and it is the order of $\mathbb{Z}_p \oplus \mathbb{Z}_q$. Thus, $\mathbb{Z}_p \oplus \mathbb{Z}_q$ is the cyclic group, generated by (a, b). $\square$