Question 1. *If A, B are subgroups of G such that $b^{-1}Ab \subset A$ for any $b \in B$, show that $AB \leq G$.*

Solution. Let $a_1b_1, a_2b_2 \in AB$, where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then

$$a_1b_1 \cdot a_2b_2 = a_1(b_1a_2b_1^{-1})(b_1b_2) = a_1((b_1^{-1})^{-1}a_2b_1^{-1})(b_1b_2) \in AB,$$

because $(b_1^{-1})^{-1}a_2b_1^{-1} \in A$ and $b_1b_2 \in B$. Furthermore, for any $ab \in AB$

$$(ab)^{-1} = b^{-1}a^{-1} = (b^{-1}a^{-1}b)b^{-1} \in AB.$$

And finally, since the identity e of G belongs both to A and to B, we conclude that $e = e \cdot e \in AB$. Thus, AB is a subgroup of G. □