Question 1. *Prove that polynomial ring of 2 independent variables over field is not a PID.*

Solution. Let $R[x,y]$ be the mentioned polynomial ring. Consider the ideal $I = \langle x, y \rangle = \{xP_1 + yP_2 \mid P_1, P_2 \in R[x,y]\}$. Suppose there is a polynomial $P \in R[x,y]$ such that $I = P \cdot R[x,y]$. Then any element of I should be divisible by P, in particular, P should divide x and y. But x and y have degree 1. Therefore, the degree of P should not exceed 1. So, $P = ax+by+c$ for some scalars $a, b, c \in R$. Since P divides x, we conclude that $b = c = 0$. But the fact that P divides y implies $a = c = 0$. Thus, $a = b = c = 0$ and so $P = 0$. Therefore, $I = \{0\}$, which is a contradiction (I contains, for example, x and y). □