1)

y''+x y=1,y(0)=y(1)=0

this differential equation has no solutions

2)

Differential equation solutions:

$$y(x) = x - \sin(x) + \tan\left(\frac{1}{2}\right)\cos(x)$$

Solve $\frac{d^2 y(x)}{dx^2} + y(x) = x$, such that $y'(0) = 0$ and $y'(1) = 0$:

The general solution will be the sum of
the complementary solution and particular solution.

Find the complementary solution by solving $\frac{d^2 y(x)}{dx^2} + y(x) = 0$:

Assume a solution will be proportional to $e^{\lambda x}$ for some constant λ.

Substitute $y(x) = e^{\lambda x}$ into the differential equation:

$$\frac{d^2}{dx^2}\left(e^{\lambda x}\right) + e^{\lambda x} = 0$$

Substitute $\frac{d^2}{dx^2}\left(e^{\lambda x}\right) = \lambda^2 e^{\lambda x}$:

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0$$

Factor out $e^{\lambda x}$:

$$\left(\lambda^2 + 1\right) e^{\lambda x} = 0$$

Since $e^{\lambda x} \neq 0$ for any finite λ, the zeros must come from the polynomial:

$$\lambda^2 + 1 = 0$$

Solve for λ:

$\lambda = i$ or $\lambda = -i$

The roots $\lambda = \pm i$ give $y_1(x) = c_1 e^{i x}$, $y_2(x) = c_2 e^{-i x}$ as solutions, where c_1 and c_2 are arbitrary constants.

The general solution is the sum of the above solutions:

$$y(x) = y_1(x) + y_2(x) = c_1 e^{i x} + c_2 e^{-i x}$$

Apply Euler's identity $e^{\alpha + i \beta} = e^{\alpha} \cos(\beta) + i e^{\alpha} \sin(\beta)$:

$$y(x) = c_1 (\cos(x) + i \sin(x)) + c_2 (\cos(x) - i \sin(x))$$

Regroup terms:

$$y(x) = (c_1 + c_2) \cos(x) + i (c_1 - c_2) \sin(x)$$

Redefine $c_1 + c_2$ as c_1 and $i (c_1 - c_2)$ as c_2, since these are arbitrary constants:

$$y(x) = c_1 \cos(x) + c_2 \sin(x)$$

Determine the particular solution to $\frac{d^2 y(x)}{dx^2} + y(x) = x$ by the method of undetermined coefficients:

The particular solution to $\frac{d^2 y(x)}{dx^2} + y(x) = x$ is of the form:

$$y_p(x) = a_1 + a_2 x$$

Solve for the unknown constants a_1 and a_2:

Compute $\frac{d^2 y_p(x)}{dx^2}$:

$$\begin{aligned} \frac{d^2 y_p(x)}{dx^2} &= \frac{d^2}{dx^2} (a_1 + a_2 x) \\ &= 0 \end{aligned}$$

Substitute the particular solution $y_p(x)$ into the differential equation:

$$\frac{d^2 y_p(x)}{dx^2} + y_p(x) = x$$

$$a_1 + a_2\, x = x$$

Equate the coefficients of 1 on both sides of the equation:

$$a_1 = 0$$

Equate the coefficients of x on both sides of the equation:

$$a_2 = 1$$

Substitute a_1 and a_2 into $y_p(x) = a_1 + a_2\, x$:

$$y_p(x) = x$$

The general solution is:

$$y(x) = y_c(x) + y_p(x) = x + c_1 \cos(x) + c_2 \sin(x)$$

Solve for the unknown constants using the initial conditions:

Compute $\frac{dy(x)}{dx}$:

$$\begin{aligned}\frac{dy(x)}{dx} &= \frac{d}{dx}(x + c_1 \cos(x) + c_2 \sin(x))\\ &= -c_1 \sin(x) + c_2 \cos(x) + 1\end{aligned}$$

Substitute $y'(0) = 0$ into $\frac{dy(x)}{dx} = -c_1 \sin(x) + c_2 \cos(x) + 1$:

$$c_2 + 1 = 0$$

Substitute $y'(1) = 0$ into $\frac{dy(x)}{dx} = -c_1 \sin(x) + c_2 \cos(x) + 1$:

$$-\sin(1)\, c_1 + \cos(1)\, c_2 + 1 = 0$$

Solve the system:

$$c_1 = -\cot(1) + \csc(1)$$
$$c_2 = -1$$

Substitute $c_1 = -\cot(1) + \csc(1)$ and $c_2 = -1$ into $y(x) = x + c_1 \cos(x) + c_2 \sin(x)$:

$$y(x) = x - \sin(x) + \tan\left(\frac{1}{2}\right) \cos(x)$$