Answer on Question #46094-Engineering-Other

a) Let a quadratic form have the expression $x^2+y^2+2z^2+2xy+3xz$ with respect to the standard basis $B_1=\{(1,0,0),(0,1,0),(0,0,1)\}$. Find its expression with respect to the basis $B_2=\{(1,1,1),(0,1,0),(0,1,1)\}$.

b) Consider the quadratic form $Q: 2x^2+y^2+3z^2-4xy+4xz$

i) Find a symmetric matrix A such that $Q=X^tAX$.

ii) Find the orthogonal canonical reduction of the quadratic form.

iii) Find the principal axes of the form.

iv) Find the rank and signature of the form.

Solution

a) $\begin{pmatrix}x\\y\\z\end{pmatrix}=x'\begin{pmatrix}1\\1\\1\end{pmatrix}+y'\begin{pmatrix}0\\1\\0\end{pmatrix}+z'\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}x'\\x'+y'+z'\\x'+z'\end{pmatrix}.$

The expression of a quadratic form with respect to the basis B_2 is

$$x'^2+(x'+y'+z')^2+2(x'+z')^2+2x'(x'+y'+z')+3x'(x'+z')=$$
$$9x'^2+y'^2+3z'^2+4x'y'+11x'z'+2y'z'.$$

b) i) $A=\begin{pmatrix}2&-2&2\\-2&1&0\\2&0&3\end{pmatrix}.$

ii) The characteristic equation for Q is $\lambda^3-D_1\lambda^2+D_2\lambda-D_3=0.$

$$D_1=2+1+3=6;$$
$$D_2=\begin{vmatrix}2&-2\\-2&1\end{vmatrix}+\begin{vmatrix}1&0\\0&3\end{vmatrix}+\begin{vmatrix}2&2\\2&3\end{vmatrix}=3.$$
$$D_3=|A|=-10.$$

Hence

$$\lambda^3-6\lambda^2+3\lambda+10=0.$$

Easy to see that $\lambda=-1$ is the root of equation.

$$\lambda^3-6\lambda^2+3\lambda+10=(\lambda+1)(\lambda^2-7\lambda+10)=(\lambda+1)(\lambda-2)(\lambda-5).$$

Therefore the orthogonal canonical reduction of the quadratic form is

$$Q=(x'\quad y'\quad z')\begin{pmatrix}-1&0&0\\0&2&0\\0&0&5\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}=-x'^2+2y'^2+5z'^2.$$

iii) $\lambda=-1$

$$\begin{pmatrix}2+1&-2&2\\-2&1+1&0\\2&0&3+1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\-2\end{pmatrix}.$$

The principal axis is $\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}.$

$$\lambda=2$$

$$\begin{pmatrix}2-2&-2&2\\-2&1-2&0\\2&0&3-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-2\\-2\end{pmatrix}.$$

The principal axis is $\frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$.

$$\lambda = 5$$

$$\begin{pmatrix} 2-5 & -2 & 2 \\ -2 & 1-5 & 0 \\ 2 & 0 & 3-5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}.$$

The principal axis is $\frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$.

v) The rank is 3 and signature is $-1 + 1 + 1 = 1$.