Answer on Question #46093-Engineering-Other

a) Check whether the forms $2x^2 + 3y^2 + 5z^2 - 4xz - 6yz$ and $4x^2 + 3y^2 + z^2 - 6xy - 2xz$ are orthogonally equivalent.

b) Use Gram-Schmidt orthogonalisation process to find an orthonormal basis for the subspace of $\mathbb{C}^4$ generated by the vectors $(1, i, 0, -i)$, $(-i, 0,1,2)$ and $(0, -i, 1,1)$.

c) Which of the following matrices are Hermitian and which are Unitary? Justify your answer.

$$A = \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 1-i \\ 0 & 1+i & 2 \end{pmatrix}, B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \sqrt{2}i & 0 & \sqrt{2}i \end{pmatrix}.$$

Solution

a) Two quadratic forms are called orthogonally equivalent, if there exists an orthogonal transformation from one to another. It is known that two quadratic forms are orthogonally equivalent if the characteristic polynomials of their matrixes are the same (since the orthogonal transformation doesn't change the characteristic polynomial of the matrix).

$$q_1 = 2x^2 + 3y^2 + 5z^2 - 4xz - 6yz, q_2 = 4x^2 + 3y^2 + z^2 - 6xy - 2xz.$$

Their matrixes are A and B, respectively:

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & -3 \\ -2 & -3 & 5 \end{pmatrix}, B = \begin{pmatrix} 4 & -3 & -1 \\ -3 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

$$P_A(t) = det(A - tE) = \begin{vmatrix} 2-t & 0 & -2 \\ 0 & 3-t & -3 \\ -2 & -3 & 5-t \end{vmatrix} = (2-t)(15 - 8t + t^2) - 2(6 - 2t)$$
$$= -t^3 + 10t^2 - 27t + 18.$$

$$P_B(t) = det(B - tE) = \begin{vmatrix} 4-t & -3 & -1 \\ -3 & 3-t & 0 \\ -1 & 0 & 1-t \end{vmatrix} = -1(3 - t) + (1 - t)(3 - 7t + t2)$$
$$= -t^3 + 8t^2 - 9t.$$

So, the equality $P_A(t) \equiv P_B(t)$ doesn't hold, hence these forms are not orthogonally equivalent and we are done.

b) $a_1 = (1, i, 0, -i), a_2 = (-i, 0,1,2), a_3 = (0, -i, 1,1).$

$b_1 = a_1 = (1, i, 0, -i).$

$b_2 = a_2 - \frac{(a_2,b_1)}{(b_1,b_1)} b_1 = (-i, 0,1,2) - (1, i, 0, -i) \cdot \frac{-i\cdot 1+0(-i)+1\cdot 0+2\cdot i}{1\cdot 1+i(-i)+0\cdot 0-i\cdot i} = (-i, 0,1,2) - (1, i, 0, -i) \cdot \frac{i}{3} = \frac{1}{3}(-4i, 1,3,5)$.

$b_3 = a_3 - \frac{(a_3,b_1)}{(b_1,b_1)} b_1 - \frac{(a_3,b_2)}{(b_2,b_2)} b_2 = (0, -i, 1,1) - (1, i, 0, -i) \cdot \frac{0\cdot 1+(-i)(-i)+1\cdot 0+1\cdot i}{1\cdot 1+i(-i)+0\cdot 0-i\cdot i} - \frac{1}{3}(-4i, 1,3,5) \cdot \frac{\frac{1}{3}(0\cdot 4i+(-i)\cdot 1+1\cdot 3+1\cdot 5)}{\frac{1}{9}(-4i\cdot 4i+1\cdot 1+3\cdot 3+5\cdot 5)} = (0, -i, 1,1) - (1, i, 0, -i) \cdot \frac{i-1}{3} - (-4i, 1,3,5) \cdot \frac{8-i}{51} = \frac{1}{17}(5i + 7, -11i + 3,9 + i, -2 - 4i)$.

$|b_1| = \sqrt{1\cdot 1 + i(-i) + 0 \cdot 0 - i \cdot i} = \sqrt{3}.$

$|b_2| = \frac{1}{3}\sqrt{-4i \cdot 4i + 1 \cdot 1 + 3 \cdot 3 + 5 \cdot 5} = \frac{\sqrt{51}}{3}.$

$|b_3| = \frac{1}{17}\sqrt{(5i+7)\cdot(-5i+7)+(-11i+3)\cdot(11i+3)+(9+i)\cdot(9-i)+(-2-4i)\cdot(-2+4i)} = \frac{\sqrt{306}}{17}$.

Vectors b_1, b_2, b_3 form an orthogonal basis. We will normalize vectors b_1, b_2, b_3.

$c_1 = \frac{b_1}{|b_1|} = \frac{1}{\sqrt{3}}(1, i, 0, -i), c_2 = \frac{b_2}{|b_2|} = \frac{1}{\sqrt{51}}(-4i, 1,3,5), c_3 = \frac{b_3}{|b_3|} = \frac{1}{\sqrt{306}}(5i+7, -11i+3, 9+i, -2-4i)$.

c) Matrix A is Hermitian, because its entries are equal to own conjugate transpose. Matrix B is not Hermitian, because conjugate transpose of $\sqrt{2}i$ is equal to $-\sqrt{2}i$, not $\frac{1}{\sqrt{2}}$. Both matrices are not Unitary, because absolute value of every determinant is not equal to one. Determinant of A is equal to -2 , determinant of B is equal to $2i$.