Answer on Question #46091-Engineering-Other

Let

$$A = \begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix}$$

a) Find the adjoint of A. Find the inverse of A from the adjoint of A. (4)

b) Find the characteristic and minimal polynomials of A. Hence find its eigenvalues and eigenvectors. (6)

c) Why is A diagonalisable? Find a matrix P such that P1AP is diagonal. (2)

d) Verify Cayley-Hamilton theorem for A. Hence, find the inverse of A

Solution

a) <u>Find the adjoint of A.</u>
First find the cofactor of each element.
$A_{11} = \begin{vmatrix} 7 & -6 \\ 12 & -11 \end{vmatrix} = -5, A_{12} = -\begin{vmatrix} 6 & -6 \\ 12 & -11 \end{vmatrix} = -6, A_{13} = \begin{vmatrix} 6 & 7 \\ 12 & 12 \end{vmatrix} = -12, A_{21} = -\begin{vmatrix} 4 & -4 \\ 12 & -11 \end{vmatrix} = -4, A_{22} = \begin{vmatrix} 5 & -4 \\ 12 & -11 \end{vmatrix} = -7, A_{23} = -\begin{vmatrix} 5 & 4 \\ 12 & 12 \end{vmatrix} = -12, A_{31} = \begin{vmatrix} 4 & -4 \\ 7 & -6 \end{vmatrix} = 4, A_{32} = -\begin{vmatrix} 5 & -4 \\ 6 & -6 \end{vmatrix} = 6, A_{33} = \begin{vmatrix} 5 & 4 \\ 6 & 7 \end{vmatrix} = 11.$

As a result the cofactor matrix of A is

$$\begin{pmatrix} -5 & -6 & -12 \\ -4 & -7 & -12 \\ 4 & 6 & 11 \end{pmatrix}.$$

Finally the adjoint of A is the transpose of the cofactor matrix:

$$\begin{pmatrix} -5 & -4 & 4 \\ -6 & -7 & 6 \\ -12 & -12 & 11 \end{pmatrix}.$$

<u>Find the inverse of A from the adjoint of A.</u>

$$\det A = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} = -1.$$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} -5 & -4 & 4 \\ -6 & -7 & 6 \\ -12 & -12 & 11 \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} -5 & -4 & 4 \\ -6 & -7 & 6 \\ -12 & -12 & 11 \end{pmatrix} = \begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix}.$$

b) <u>The characteristic equation</u> for Q is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0.$

$$D_1 = 5 + 7 - 11 = 1;$$
$$D_2 = A_{11} + A_{22} + A_{33} = -5 - 7 + 11 = -1.$$
$$D_3 = |A| = -1.$$

Hence

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0 \rightarrow (\lambda - 1)^2(\lambda + 1) = 0.$$

<u>The minimal polynomials of A are</u>

$$(\lambda - 1), (\lambda - 1)(\lambda + 1), (\lambda + 1), (\lambda - 1)^2.$$

<u>The eigenvalues of A</u> are $\lambda_1 = -1$ and $\lambda_2 = +1$.
<u>Find eigenvector corresponding</u> $\lambda_1 = -1$

$$\begin{pmatrix} 5+1 & 4 & -4 \\ 6 & 7+1 & -6 \\ 12 & 12 & -11+1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to v_1 = \frac{1}{7}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}.$$

<u>Find eigenvectors corresponding</u> $\lambda_2 = 1$

$$\begin{pmatrix} 5-1 & 4 & -4 \\ 6 & 7-1 & -6 \\ 12 & 12 & -11-1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to x + y - z = 0 \to v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

c) <u>Because A have eigenvector basis.</u>
<u>Find a matrix P such that $P^{-1}AP$ is diagonal.</u>

$$\mathrm{P} = \begin{pmatrix} \frac{2}{7} & \frac{1}{\sqrt{2}} & 0 \\ \frac{3}{7} & 0 & \frac{1}{\sqrt{2}} \\ \frac{6}{7} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

$$P^{-1}AP = D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

d) <u>Verify Cayley-Hamilton theorem for A. Hence, find the inverse of A.</u>
We have to show that

$$A^3 - A^2 - A + I = 0.$$

$$A^2 = \begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix}\begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$A^3 = A^2A = IA = A.$$

So

$$A - I - A + I = 0.$$

This verifies Cayley-Hamilton theorem.

The inverse of A:

$$A^2 = AA = 1 \to A^{-1} = A = \begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix}.$$