Task 1. Let $R$ be a commutative ring with unity and let $a, b \in R$. Prove that if $ab$ has a multiplicative inverse in $R$, then both $a$ and $b$ have multiplicative inverses.

**Proof.** Let $c$ be the multiplicative inverse of $ab$, so $cab = 1$. Then

$$(ca)b = 1,$$

and so

$$ca = b^{-1}$$

is the inverse of $b$.

Similarly, since $R$ is commutative, $ab = ba$, and so

$$cba = cab = 1.$$

Thus

$$(cb)a = 1,$$

and therefore

$$cb = a^{-1}$$

is the inverse of $a$.

Task 2. Let $R = 2\mathbb{Z}$ be the ring of even integers. Show that $R$ contains a maximal ideal $M$ so that $R/M$ is not a field.

**Proof.** Let $M = 4\mathbb{Z} \subseteq R$ be the ring of integers which are multiples of 4. We claim that $R/M$ is not a field.

We will prove that $R/M$ has zero divisors. Indeed, let $[2] = 2 + M$ be the class of 2 in $R/M$, and $[0] = M$ be the class of 0 in $R/M$. Then


since $4 \in M$.

Thus $[2]$ is a zero divisor in $R/M$, and so $R/M$ is not a field.

Task 3. Prove that if $R$ is a commutative ring with unity and $f = a_nx^n + \cdots + a_0$ is a zero divisor in $R[x]$, then there exists a nonzero $b$ in $R$ such that

$$ba_n = b^2a_{n-1} = b^3a_{n-2} = \cdots = 0.$$

**Proof.** The assumption that $f$ is a zero divisor in $R[x]$ means that there exists a polynomial $g = b_mx^m + b_{m-1}x^{m-1} + \cdots + b_0$ such that $b_m \neq 0$ and $gf = 0$ in $R[x]$.

We claim that the coefficient $b_m$ at $x_m$ has the required property:

$$b_ma_n = b_m^2a_{n-1} = b_m^3a_{n-2} = \cdots = 0.$$

Indeed, $fg = 0$ means that all the coefficients of $fg$ are zero. Let us write exact formulas for $fg$:

$$gf = \left( b_mx^m + b_{m-1}x^{m-1} + \cdots + b_0 \right) \cdot \left( a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 \right) =
\left( b_ma_n \right) x^{n+m} +
\left( b_na_{n-1} + b_{m-1}a_n \right) x^{n+m-1} + \cdots.$$
Thus

\[ b_{m}a_{n} = 0, \]
\[ b_{m}a_{n-1} + b_{m-1}a_{n} = 0, \]
\[ b_{m}a_{n-2} + b_{m-1}a_{n-1} + b_{m-2}a_{n} = 0, \]

and so on.

The first equation is what we need: \( b_{m}a_{n} = 0 \).

Multiplying the second equation by \( b_{m} \) we get:

\[ 0 = b_{m}(b_{m}a_{n-1} + b_{m-1}a_{n}) = b_{m}^{2}a_{n-1} + b_{m}b_{m-1}a_{n} = b_{m}^{2}a_{n-1} + b_{m-1}(b_{n}a_{n}) = \]
\[ = b_{m}^{2}a_{n-1} + b_{m-1} \cdot 0 = b_{m}^{2}a_{n-1}, \]

Thus

\[ b_{m}^{2}a_{n-1} = 0. \]

Again, multiplying the third equation by \( b_{m}^{2} \) we obtain

\[ b_{m}^{2}(b_{m}a_{n-2} + b_{m-1}a_{n-1} + b_{m-2}a_{n}) = b_{m}^{3}a_{n-2} + b_{m-1}(b_{m}^{2}a_{n-1}) + b_{m-2}b_{m}(b_{n}a_{n}) = \]
\[ = b_{m}^{3}a_{n-2} + b_{m-1} \cdot 0 + b_{m-2}b_{m} \cdot 0 = b_{m}^{3}a_{n-2}. \]

Thus

\[ b_{m}^{3}a_{n-2} = 0. \]

By similar arguments multiplying coefficient at \( x^{m+n-k} \) by \( b_{m}^{k} \) we will get that

\[ b_{m}^{k}a_{n-k+1} = 0 \]

for all \( k \).